

Perfect Crystals for Quantum Affine Algebras and Combinatorics of Young walls

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$$A_\infty = \{f/g \in \mathbb{Q}(q) \mid f/g \text{ is regular at } \infty\}$$

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Definition

Let

$$E = V^{\mathbb{A}} \cap L \cap L_{\infty}.$$

Then $(V^{\mathbb{A}}, L, L_{\infty})$ is a **balanced triple** for V if

- 1 $\mathbb{A} \otimes_{\mathbb{Q}} E \cong V^{\mathbb{A}},$
- 2 $\mathbb{A}_0 \otimes_{\mathbb{Q}} E \cong L,$
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Note $\mathbb{Q}(q) \otimes_{\mathbb{Q}} E \cong V$.

Proposition The following are equivalent.

- ① $(V^{\mathbb{A}}, L, L_{\infty})$ is a balanced triple.
- ② $E \rightarrow L/qL$ is an isomorphism.
- ③ $E \rightarrow L_{\infty}/q^{-1}L_{\infty}$ is an isomorphism.

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Definition Suppose $(V^{\mathbb{A}}, L, L_{\infty})$ is a balanced triple for V , and let

$$G : L/qL \longrightarrow E$$

be the inverse of $E \xrightarrow{\sim} L/qL$.

- 1 A $\mathbb{Q}$ -basis B of L/qL is called a **local basis** of V at $q = 0$.
- 2 In this case, $G(B)$ is called the **global basis** of V corresponding to B .

$$\begin{array}{ccc}
 E = V^{\mathbb{A}} \cap L \cap L_{\infty} & \begin{array}{c} \xrightarrow{q=0} \\ \xleftarrow{G} \end{array} & L/qL \\
 \cup & & \cup \\
 G(B) & \longleftrightarrow & B
 \end{array}$$

2. Quantum Groups

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$$a_{ii} = 2 \quad \forall i \in I, \quad a_{ij} \leq 0 \quad i \neq j, \quad a_{ij} = 0 \Leftrightarrow a_{ji} = 0$$

$$\exists D = \text{diag}(s_i \in \mathbf{Z}_{>0} | i \in I) \text{ s.t. } DA \text{ is symmetric}$$

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$P = \{\lambda \in \mathfrak{h}^* | \lambda(P^\vee) \subset \mathbb{Z}\}$, where $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$

(weight lattice)

$$\Pi = \{ \alpha_i \in \mathfrak{h}^* | i \in I \}, \quad \text{where} \quad \alpha_j(h_i) = a_{ij}, \quad \alpha_j(d_s) = 0 \text{ or } 1$$

(simple roots)

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 $=$ Lie algebra over $\mathbb{C}$ generated by $e_i, f_i (i \in I), h \in P^\vee$
 with defining relations

- i) $[h, h'] = 0$ for $h, h' \in P^\vee$
- ii) $[e_i, f_j] = \delta_{ij} h_i$
- iii) $[h, e_i] = \alpha_i(h) e_i, [h, f_i] = -\alpha_i(h) f_i$ for $h \in P^\vee$
- iv) $(ade_i)^{1-a_{ij}}(e_j) = (adf_i)^{1-a_{ij}}(f_j) = 0$ for $i \neq j$

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3 Kac-Moody algebras of indefinite type

$\mathcal{O}_{int}$ = the category consists of $\mathfrak{g}$ -modules M such that

- i) $M = \bigoplus_{\lambda \in P} M_{\lambda}$, where $M_{\lambda} = \{v \in M \mid hv = \lambda(h)v \quad \forall h \in P^{\vee}\}$,
- ii) $\exists \lambda_1, \lambda_2, \dots, \lambda_s \in P$ such that

$$wt(M) \subset \bigcup_{i=1}^s (\lambda_i - Q_+),$$

where $Q_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$,

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The **character** of a weight module M is

$$\text{ch} M = \sum_{\mu} (\dim_{\mathbb{C}} M_{\mu}) e^{\mu}.$$

Theorem Let $V(\lambda)$ be the irreducible height weight $U(\mathfrak{g})$ -module with height weight $\lambda \in \mathfrak{h}^*$. Set

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- ③ $\mathcal{O}_{int}$ is semisimple.

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= associative algebra over $\mathbb{C}(q)$ generated by
 e_i, f_i ($i \in I$), q^h ($h \in P^\vee$) with defining relations :

- i) $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$
- ii) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in P^\vee$
- iii) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$

$$\text{iv)} \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad \text{for } i \neq j$$

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Here, $q_i = q^{s_i}$ and $K_i = q^{s_i h_i}$ and $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$,

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

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Theorem (Drinfeld 1985, Jimbo 1985, Lusztig 1990, ...)

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$$\left\{ \begin{array}{c} \text{repns of } U_q(\mathfrak{g}) \\ \text{in } \mathcal{O}_{int}^q \end{array} \right\} \xrightarrow{q=1} \left\{ \begin{array}{c} \text{repns of } U(\mathfrak{g}) \\ \text{in } \mathcal{O}_{int} \end{array} \right\}$$



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1-point functions (mathematical physics), etc.

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$$q = 0 !!$$

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$$M \in \mathcal{O}_{int}^q, \quad i \in I$$

$\implies$ Every $u \in M_\lambda (\lambda \in P)$ has an **i-string decomposition** :

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Define the **Kashiwara operator** $\tilde{e}_i, \tilde{f}_i : M \rightarrow M$ by

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.$$

Definition $M \in \mathcal{O}_{int}^q$

A **crystal basis** of M is a pair (L, B) , where

- i) L is a free $\mathbb{A}_0$ - submodule of M s.t. $\mathbb{Q}(q) \otimes_{\mathbb{A}_0} L \cong M$
- ii) $B = \mathbb{Q}$ -basis of $L/qL \cong \mathbb{Q} \otimes_{\mathbb{A}_0} L$
- iii) $L = \bigoplus_{\lambda \in P} L_\lambda$, $L_\lambda = L \cap M_\lambda$
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Remark “Most of” combinatorial features of M are reflected on its crystal graph B , e.g., $\dim_{\mathbb{Q}(q)} M_\lambda = \#B_\lambda$.

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$$\left\{ \begin{smallmatrix} x \\ t \end{smallmatrix} \right\}_{q_i} = \frac{1}{[t]_{q_i}!} \prod_{k=1}^t \frac{xq^{1-k} - x^{-1}q^{-1+k}}{q_i - q_i^{-1}}.$$

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Let

$L(\lambda) \stackrel{\text{def}}{=} \text{a free } \mathbb{A}_0\text{-submodule of } V^q(\lambda) \text{ spanned by } \widetilde{f}_{i_1} \dots \widetilde{f}_{i_r} v_\lambda (r \geq 0, i_k \in I),$

$B(\lambda) \stackrel{\text{def}}{=} \{ \widetilde{f}_{i_1} \dots \widetilde{f}_{i_r} v_\lambda + qL(\lambda) \mid r \geq 0, i_k \in I \} \setminus \{0\} \subset L(\lambda)/qL(\lambda).$

— : $U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ is the $\mathbb{Q}$ -algebra automorphism defined by

$$\bar{q} = q^{-1}, \quad \bar{e}_i = e_i, \quad \bar{f}_i = f_i, \quad \bar{q}^h = q^{-h} \quad (h \in P^\vee, i \in I).$$

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Theorem (Kashiwara 1991, cf. Lusztig 1990)

- ① $(V^\mathbb{A}(\lambda), L(\lambda), \overline{L(\lambda)})$ is a balanced triple of $V(\lambda)$
- ② $(L(\lambda), B(\lambda))$ is a crystal basis of $V(\lambda)$

Recall

$$\begin{array}{ccc}
 E = V^{\mathbb{A}}(\lambda) \cap L(\lambda) \cap \overline{L(\lambda)} & \begin{array}{c} \xrightarrow{q=0} \\ \xleftarrow{G} \end{array} & L(\lambda)/qL(\lambda) \\
 \cup & & \cup \\
 G(\lambda) & \longleftrightarrow & B(\lambda)
 \end{array}$$

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Problem

- 1 How to realize $B(\lambda)$?

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Problem

- ① How to realize $B(\lambda)$?
- ② How to construct $G(\lambda)$?

Example

1 Kashiwara-Nakashima (1994)

$$\mathfrak{g} = A_{n-1} = \mathfrak{sl}_n$$

$$\lambda \in P^+ \rightsquigarrow \lambda = \text{partition of } n$$

$$B(\lambda) \cong \{\text{semistandard tableaux of shape } \lambda\}$$

2 Kashiwara-Saito, Saito (1997, 2002)

$$B(\lambda) \cong \{\text{irreducible components of certain Lagrangian subvarieties of Nakajima's quiver varieties}\}$$

For generalized Kac-Moody algebras,

① **K.-Schiffmann** (2006)

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1 K.-Schiffmann (2006)

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2 K.-Kashiwara-Schiffmann (2009)

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4. Perfect Crystals

$U_q(\mathfrak{g})$: quantum affine algebra associated with
 $(A, P^\vee, \Pi^\vee, P, \Pi)$ (affine Cartan datum)

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$$\bar{P}_\ell^+ = \{\lambda \in \bar{P} \mid \langle c, \lambda \rangle = \ell, \lambda(h_i) \geq 0 \ \forall i \in I\},$$

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where c is the **canonical central element**.

$U'_q(\mathfrak{g}) = \langle e_i, f_i, K_i^{\pm 1} \mid i \in I \rangle \subset U_q(\mathfrak{g})$: quantum affine algebra
 associated with $(A, \bar{P}^\vee, \Pi^\vee, \bar{P}, \Pi)$ (**classical Cartan datum**)

For an element b in a finite $U'_q(\mathfrak{g})$ -crystal B ,

$$\varepsilon(b) := \sum \varepsilon_i(b) \Lambda_i, \quad \varphi(b) := \sum \varphi_i(b) \Lambda_i.$$

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Definition A finite $U'_q(\mathfrak{g})$ -crystal B is a **perfect crystal** of level $\ell > 0$ if

- 1 $B \otimes B$ is connected,
- 2 $\exists \lambda_0 \in \bar{P}$ such that

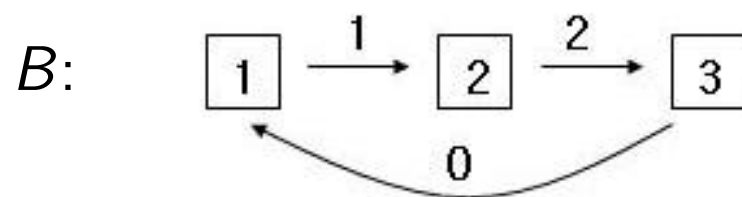
$$wt(B) \subset \lambda_0 + \frac{1}{d_0} \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i, \quad \#(B_{\lambda_0}) = 1,$$

- ③ $\langle c, \varepsilon(b) \rangle \geq \ell \quad \forall b \in B,$
- ④ for each $\lambda \in \bar{P}_\ell^+$ there exist unique $b^\lambda, b_\lambda \in B$ such that

$$\varepsilon(b^\lambda) = \lambda = \varphi(b_\lambda)$$

Example

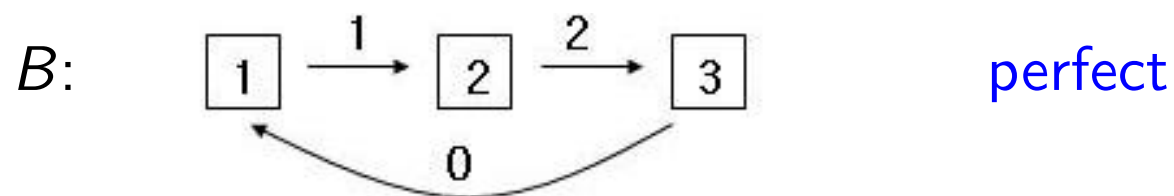
① $\mathfrak{g} = A_2^{(1)}$



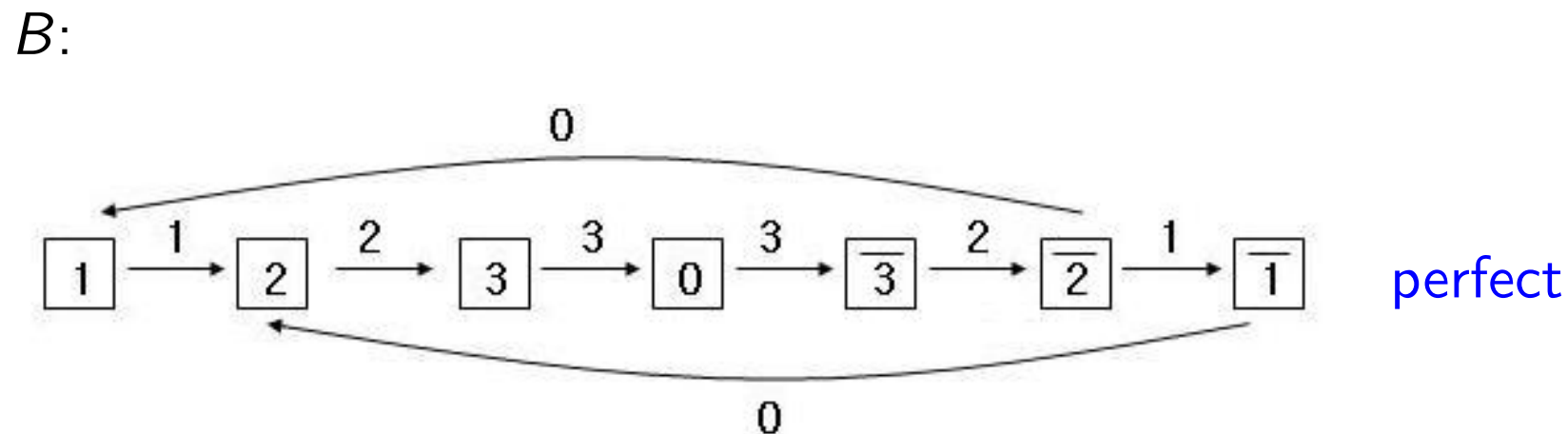
perfect

Example

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② $\mathfrak{g} = B_3^{(1)}$



Crystal Bases

oooooooooooooooooooooooooooo

Perfect Crystals

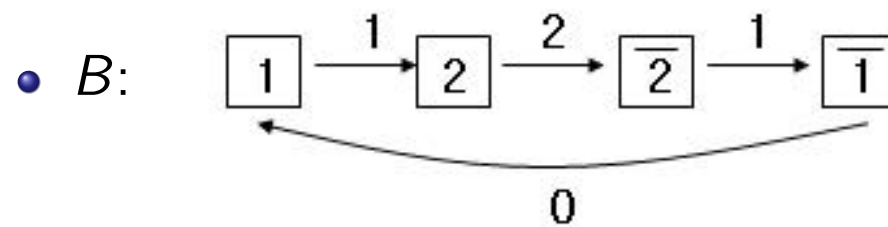
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Young Walls

oooooooooooooooooooooooooooo

Perfect Crystals

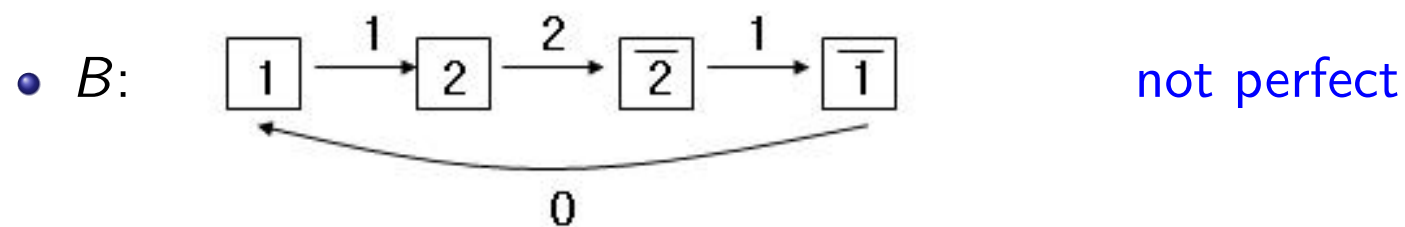
③ $\mathfrak{g} = C_2^{(1)}$



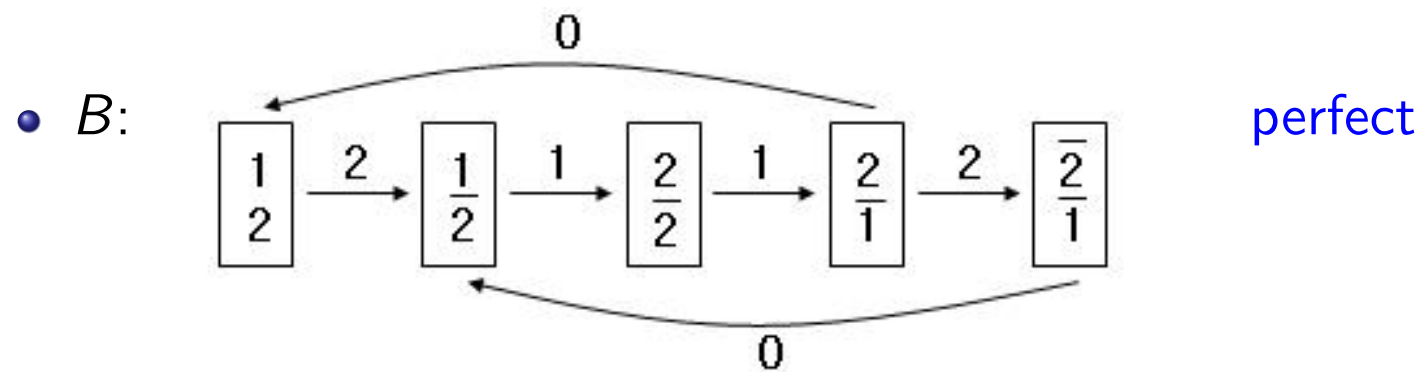
not perfect

$$b^{\Lambda_1} = [2], [\overline{1}]$$

③ $\mathfrak{g} = C_2^{(1)}$



$$b^{\Lambda_1} = \boxed{2}, \boxed{\overline{1}}$$



Theorem $((KMN)^2$ 1992)

$$B(\lambda) \simeq B(\varepsilon(b_\lambda)) \otimes B,$$

$$u_\lambda \mapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda,$$

where b_λ is the unique element of B such that $\varphi(b_\lambda) = \lambda$

Path realization

$$B(\lambda) \simeq B(\lambda_1) \otimes B \simeq B(\lambda_2) \otimes B \otimes B \simeq \dots$$

$$u_\lambda \mapsto u_{\lambda_1} \otimes b_0 \mapsto u_{\lambda_2} \otimes b_1 \otimes b_0 \mapsto \dots$$

$$\lambda_0 = \lambda, \quad b_0 = b_{\lambda_0},$$

$$\lambda_{k+1} = \varepsilon(b_{\lambda_k}), \quad b_{k+1} = b_{\lambda_{k+1}} \quad (k \geq 0)$$

$$\mathfrak{p}_\lambda = (b_k)_{k \geq 0} = \dots \otimes b_{k+1} \otimes b_k \otimes \dots \otimes b_0 \text{ (ground-state path)}$$

$\mathfrak{p} = (p_k)_{k \geq 0}$ is a λ -path in B if $p_k \in B$, $p_k = b_k \ \forall k \gg 0$

$$\mathcal{P}(\lambda) = \{ \lambda\text{-paths} \}$$

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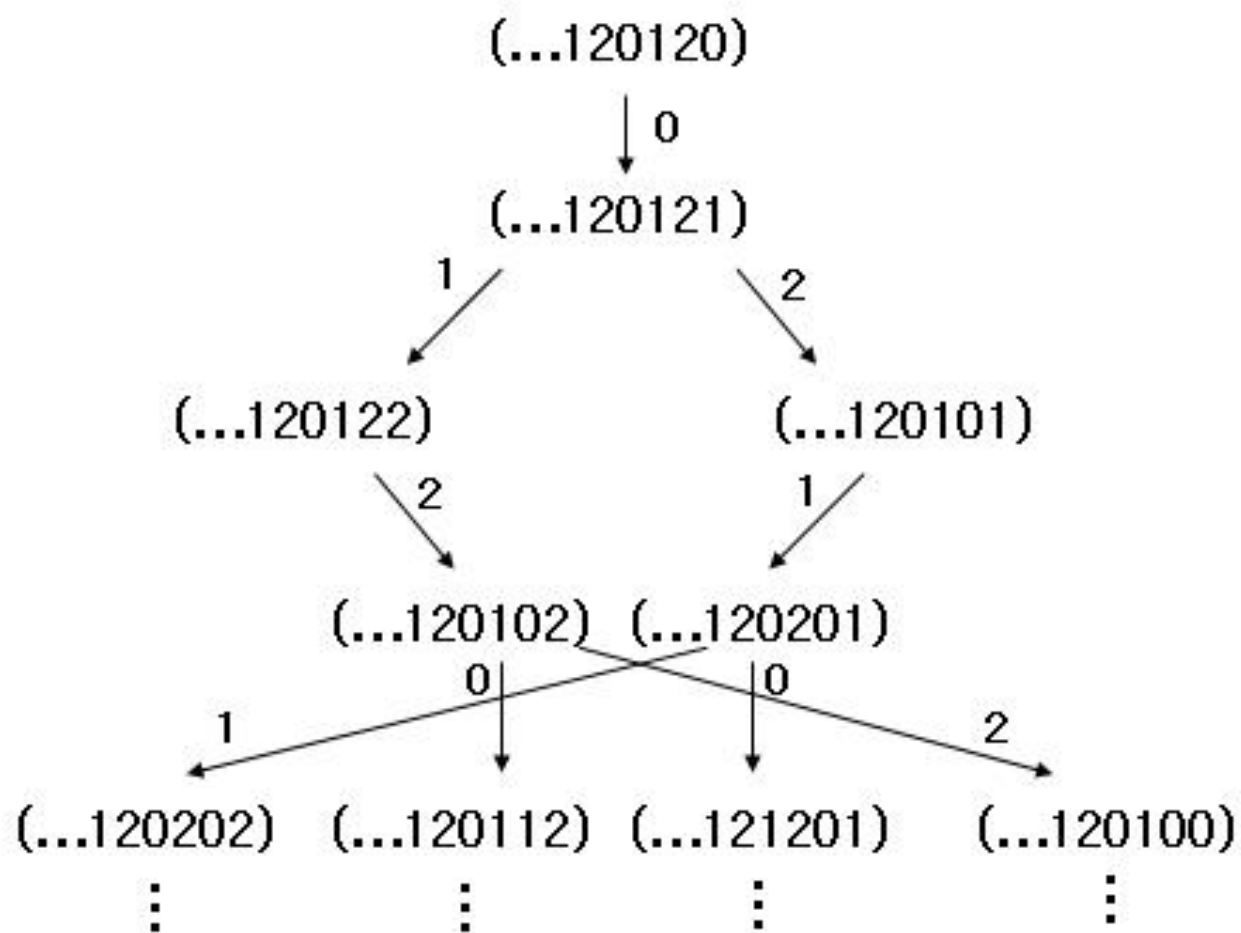
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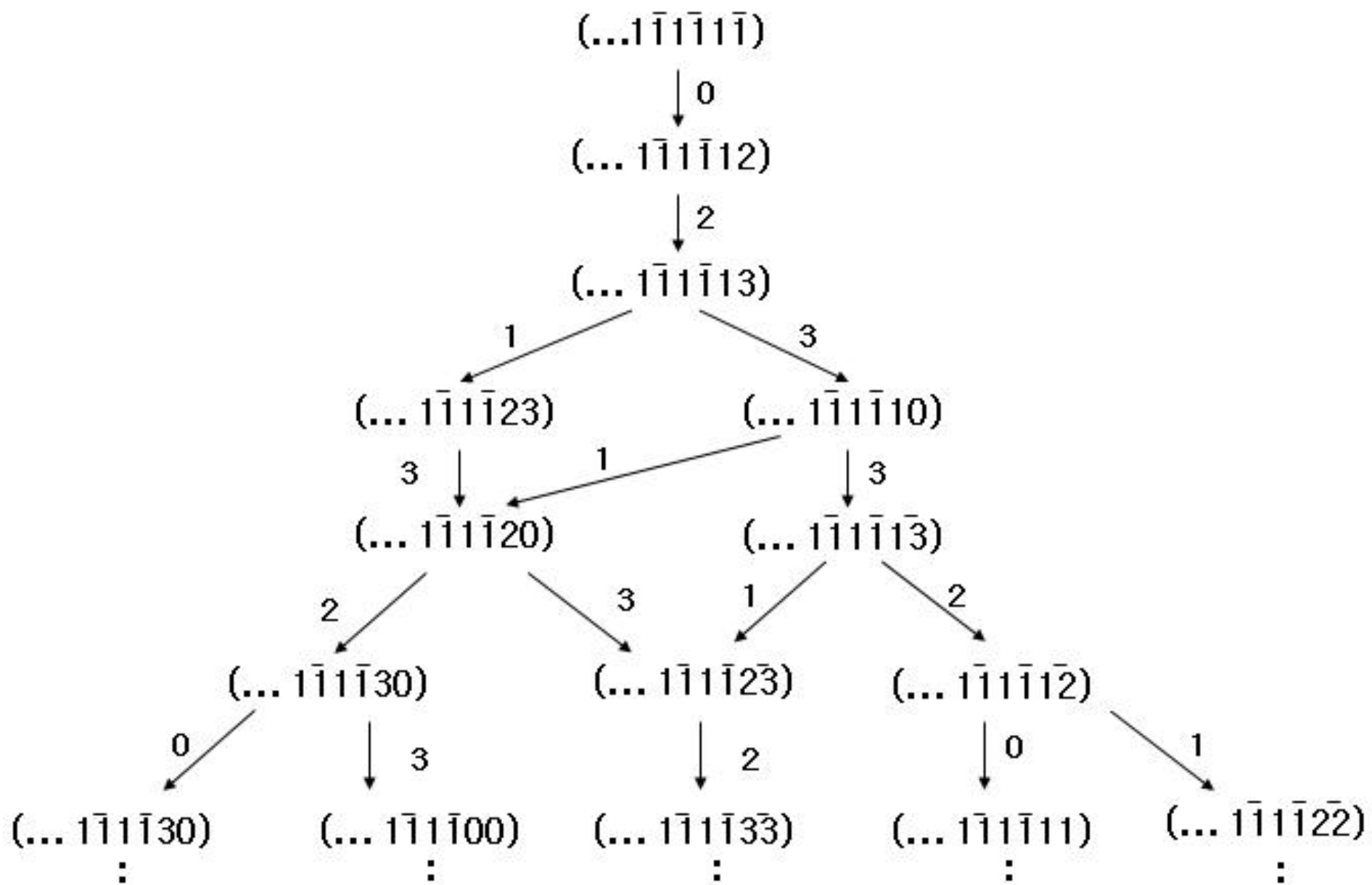
2

$$\text{ch} V(\lambda) = \sum_{\mathfrak{p} \in \mathcal{P}(\lambda)} e^{wt \mathfrak{p}}$$

Example $\mathfrak{g} = A_2^{(1)}$



Example $\mathfrak{g} = B_3^{(1)}$



Application to physics

The theory of vertex modules can be explained in the language of quantum affine algebra and crystal bases.

In particular,

$$\text{1-point function} = \frac{\text{string function}}{\text{ch } V(\lambda)}.$$

(probability for a configuration)

5.Young Walls

Let $\mathfrak{g} = A_{n-1}^{(1)}$, $\lambda = \Lambda_0$.

$$Y = \text{colored Young Diagram on } \Lambda_0$$

						1
					$n-1$	0
				.	.	$n-1$
			2	.	.	
		.	1	2	.	
	.	.	0	1	2	
.	.	.	.	$n-1$	0	1
.	.	.	.	$n-2$	$n-1$	0

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A 3x3 grid with the top-right cell shaded gray.

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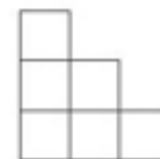
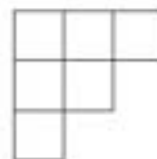
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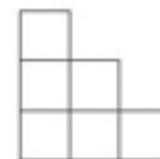
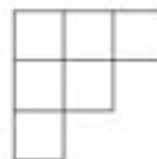
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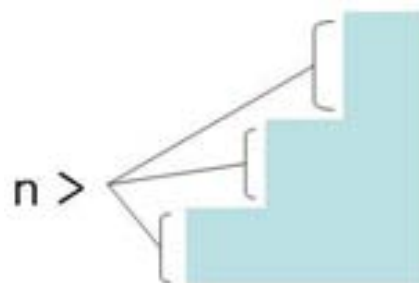
							1
						n-1	0
				.	.	n-1	
			2	.	.		
		.	1	2	.		
	.	.	0	1	2		
.	.	.	.	n-1	0	1	
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cf)



Y is n -reduced if $\ell_{k+1} - \ell_k < n \quad \forall k \geq 0$, i.e.,

n -reduced $\iff$



Let

$$\mathcal{Z}(\Lambda_0) = \{ \text{colored Young diagrams on } \Lambda_0 \},$$

$$\mathcal{Y}(\Lambda_0) = \{ n\text{-reduced Young diagrams on } \Lambda_0 \},$$

$$\mathcal{F}(\Lambda_0) = \bigoplus_{Y \in \mathcal{Z}(\Lambda_0)} \mathbb{Q}(q) Y.$$

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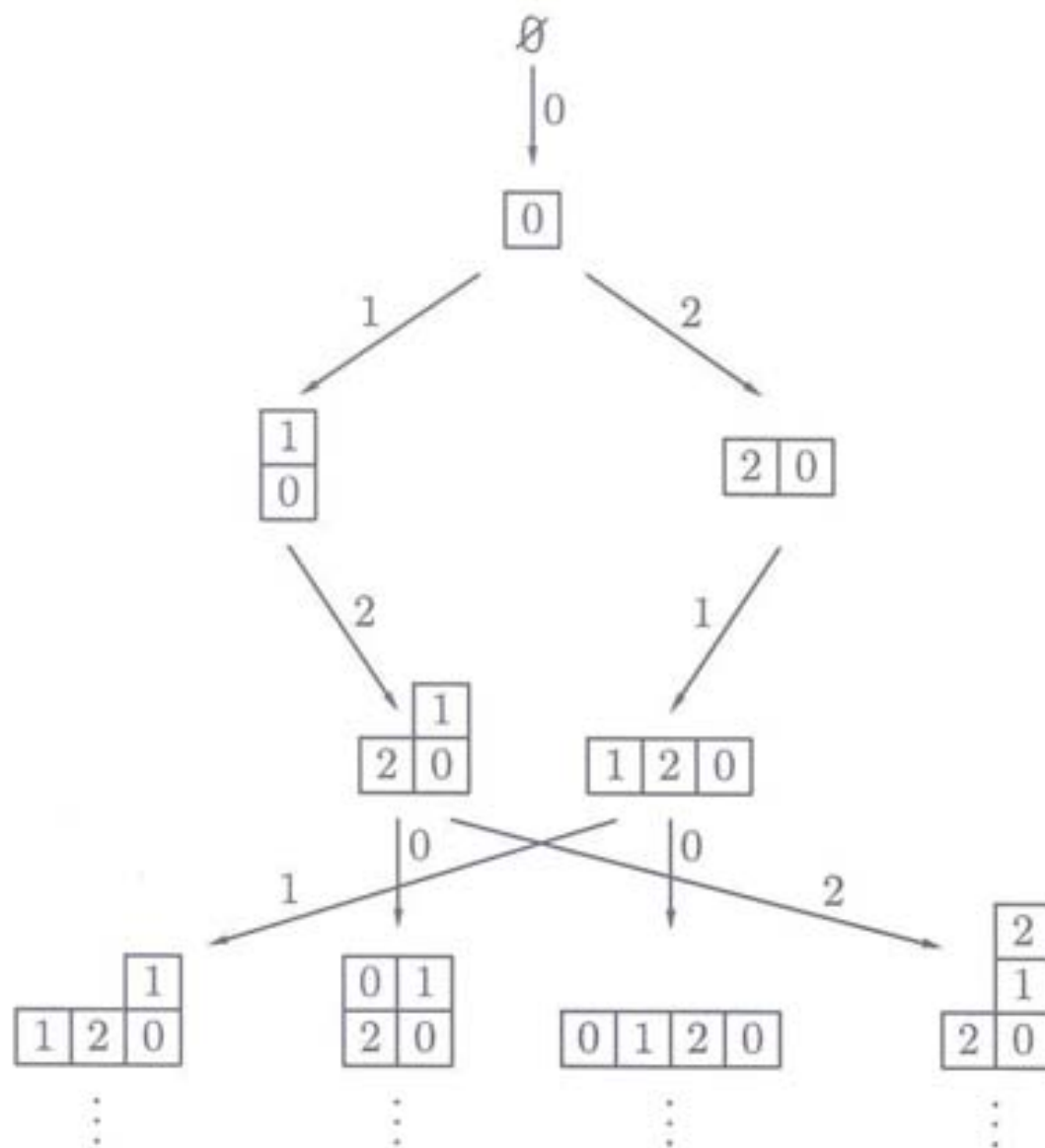
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Example The crystal $\mathcal{Y}(\Lambda_0)$ for $A_2^{(1)}$



Note $H_N(\zeta)$: **Hecke algebra** of type A with $\zeta^n = 1$.

$$\Rightarrow \{ \text{irreducible } H_N(\zeta)\text{-modules} \}$$
$$= \{D^Y \mid Y : n\text{-reduced Young diagrams with } N \text{ boxes}\}$$

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Lascoux-Leclerc-Thibon (1996)

(i) found an algorithm of computing

$$G(Y) = Y + \sum_{Y' \in \mathcal{Z}(\Lambda_0), Y' \triangleleft Y} K_{Y,Y'}(q) Y',$$

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Ariki (1996) proved (ii).

Actually, Ariki proved more:

$$B(\Lambda_0) \cong \bigoplus_{N=0}^{\infty} K_0(H_N(\zeta))$$

$$B(\lambda) \cong \bigoplus_{N=0}^{\infty} K_0(H_N^\lambda(\zeta))$$

$$B(\infty) \cong \bigoplus_{N=0}^{\infty} K_0(\widehat{H}_N(\zeta))$$

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Answer $\mathfrak{g}$: classical quantum affine algebras, $\lambda = \text{level } 1$

$\Rightarrow$ LHS is completed.

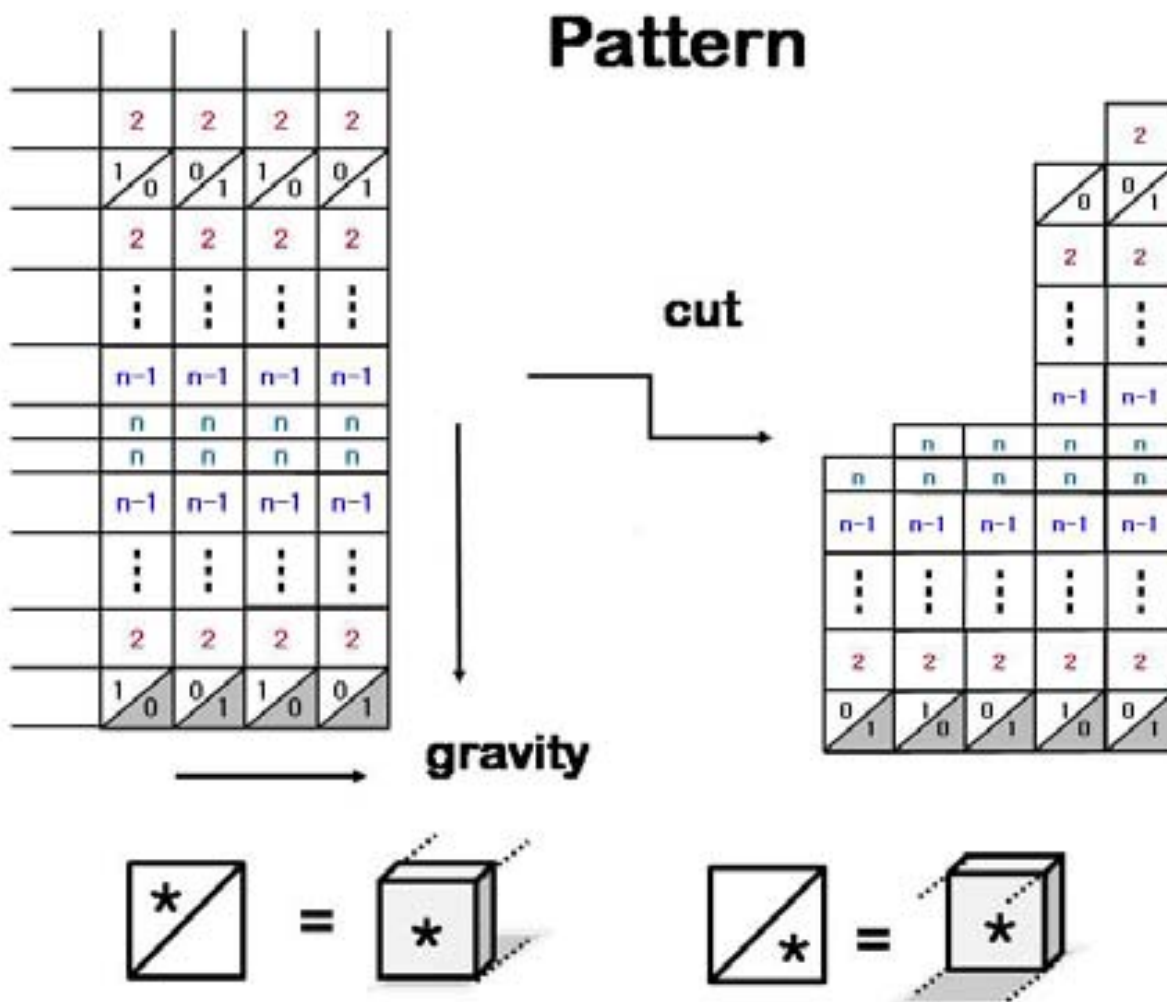
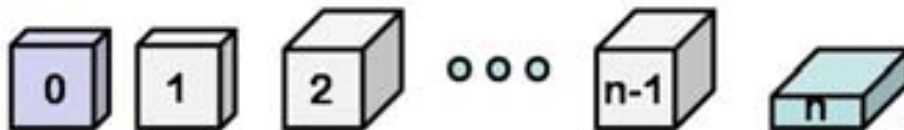
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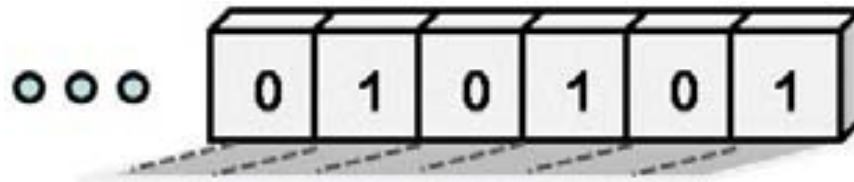
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idea: **LEGO + Tetris !!**

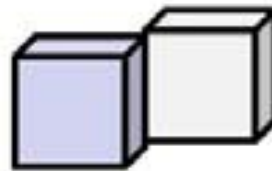
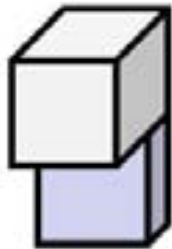
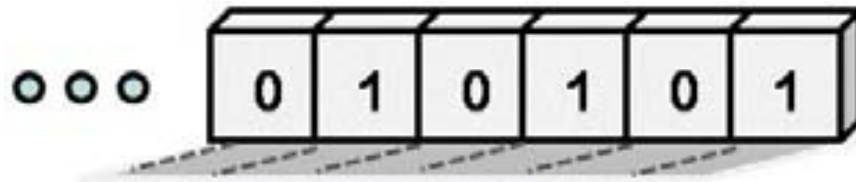
$$\mathfrak{g} = B_n^{(1)}, \quad \lambda = \Lambda_0$$



ground-state wall:



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not allowed

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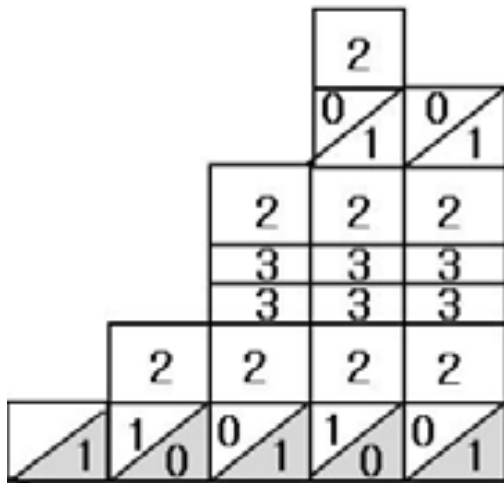
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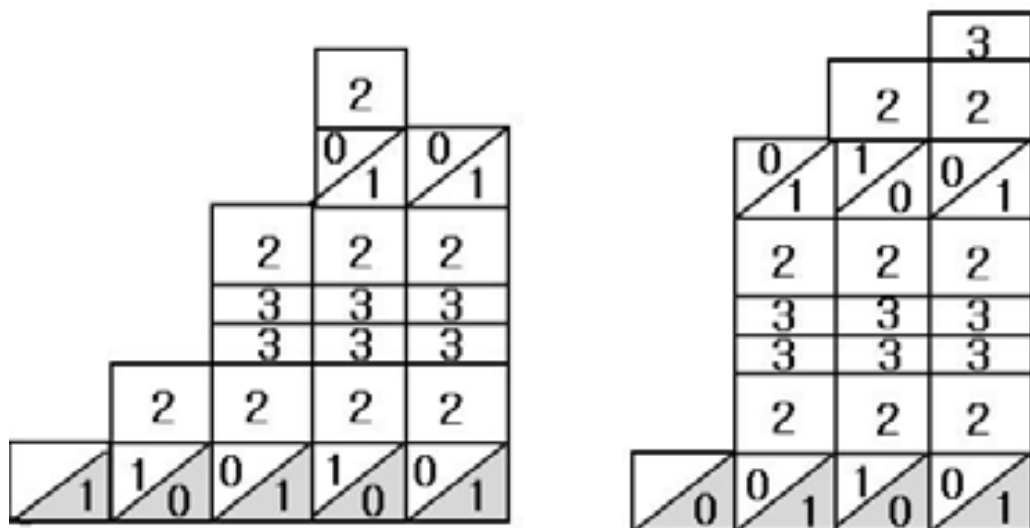
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- ⑥ A proper Young wall Y is **reduced** if $\exists$ no removable δ -column.

Example $\mathfrak{g} = B_3^{(1)}$



not a Young wall

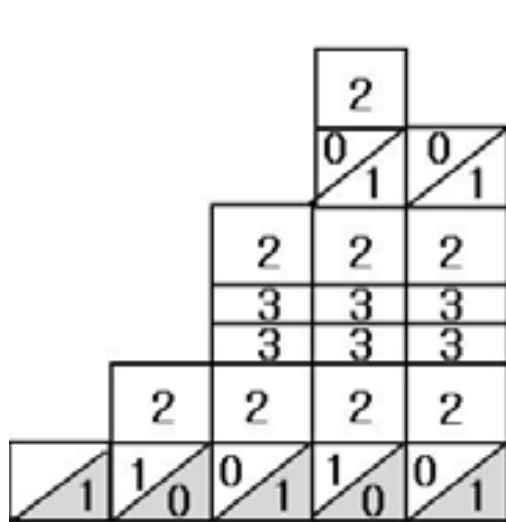
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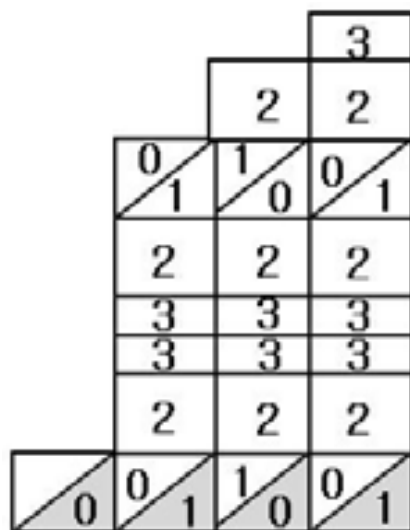
not a Young wall

proper
not reduced

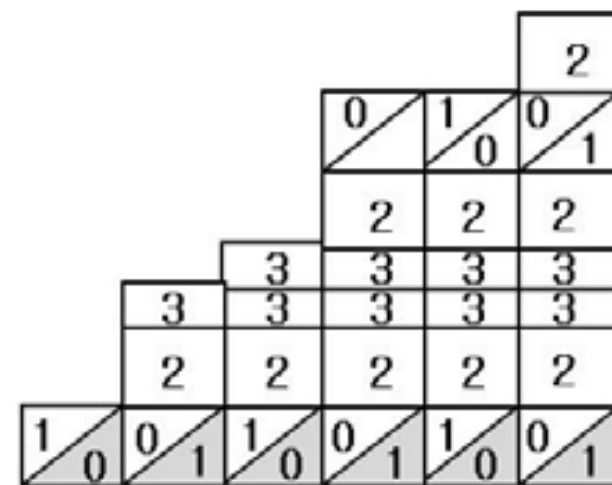
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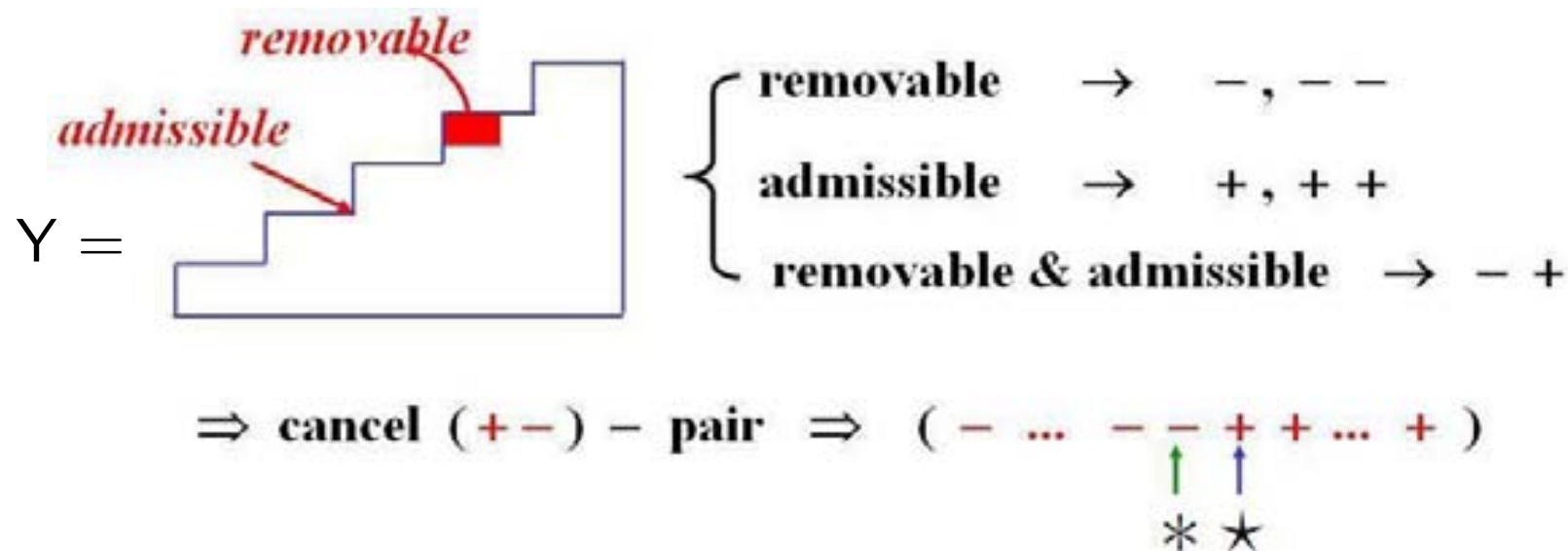
not a Young wall



proper
not reduced



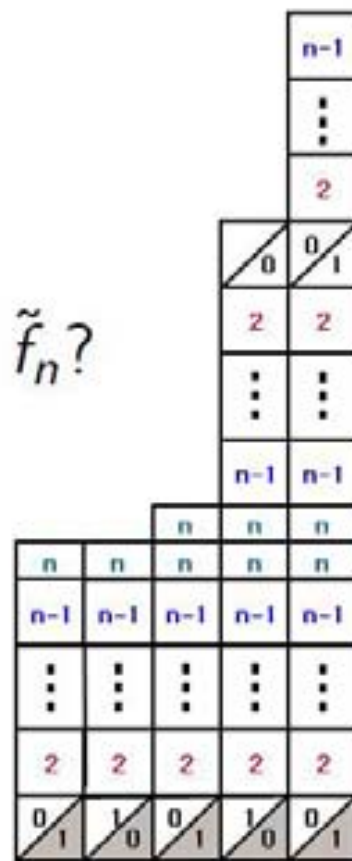
proper
reduced

$$\lambda = \Lambda_0$$


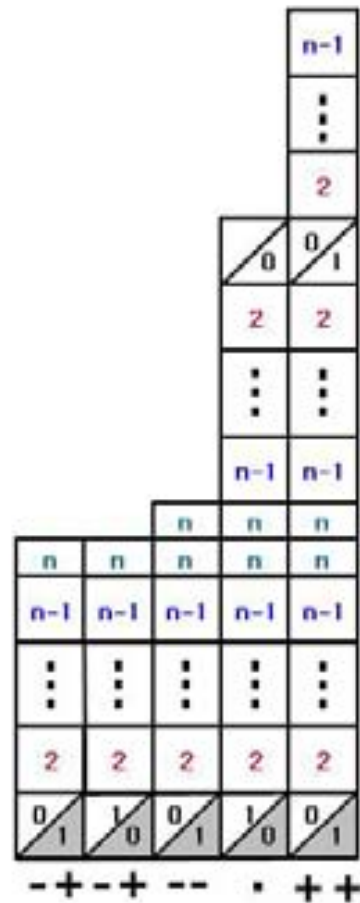
$$\begin{aligned} \tilde{e}_i Y &= Y \nearrow \boxed{i} \text{ at } *, & \tilde{f}_i Y &= Y \swarrow \boxed{i} \text{ at } \star, \\ \varepsilon_i(Y) &= \# \text{ of } -\text{'s}, & \varphi_i(Y) &= \# \text{ of } +\text{'s}, \\ \text{wt}(Y) &= \Lambda_0 - \sum_{i=0}^n k_i \alpha_i, \end{aligned}$$

where k_i is the number of i -colored blocks in Y .

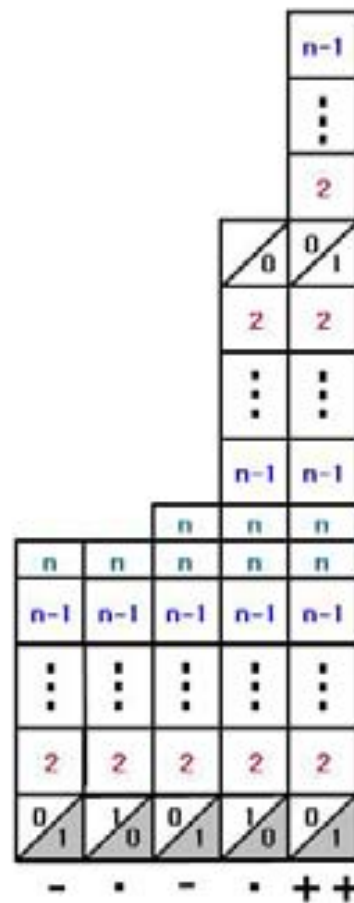
Example



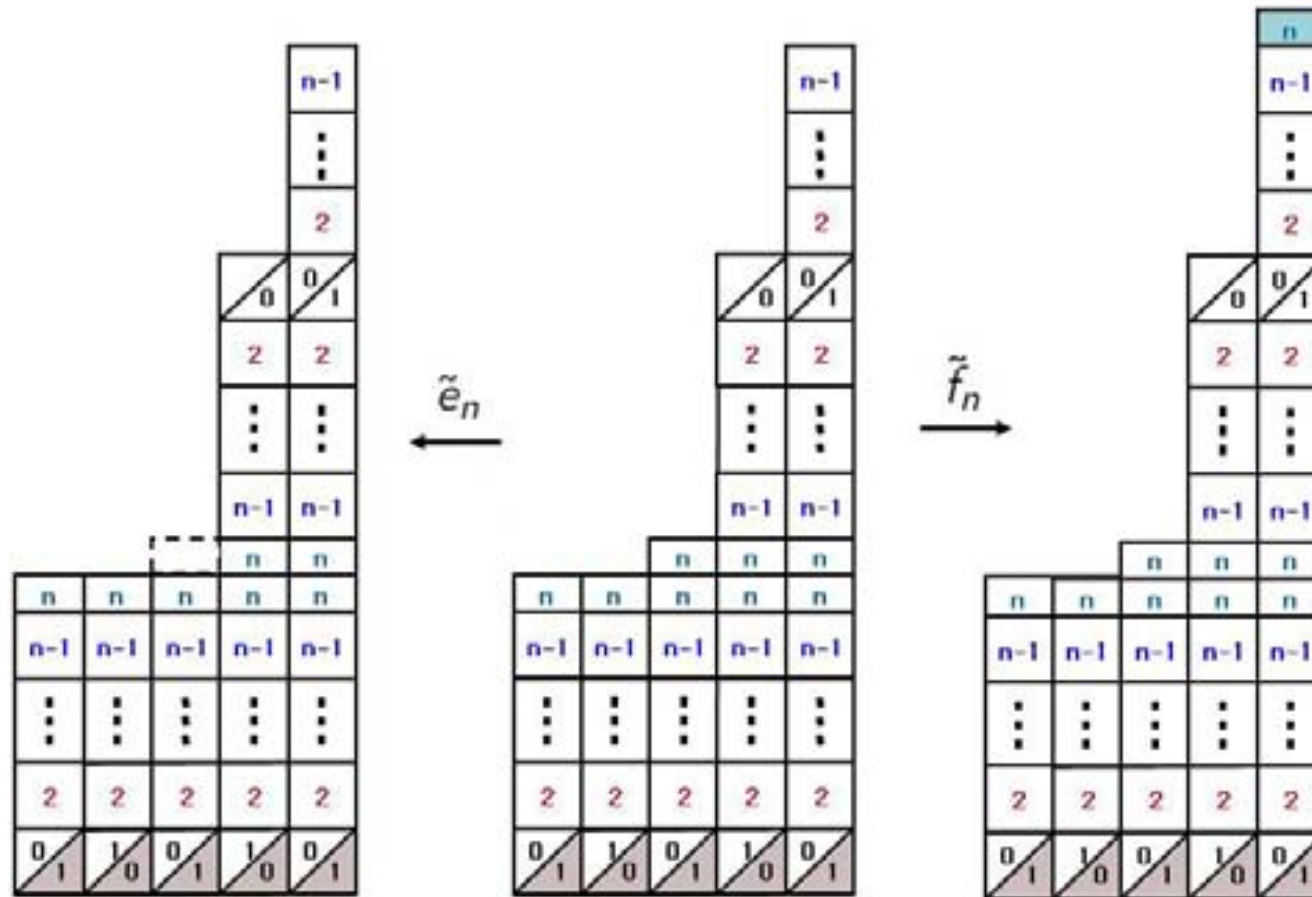
Example



Example



Example



$$\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$$

 $\lambda \in P^+ : \text{level } 1$
$$\mathcal{Z}(\lambda) = \{ \text{Young walls on } \lambda \}$$
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Theorem (K. 2003)

- ① $\mathcal{Z}(\lambda)$ is a $U_q(\mathfrak{g})$ -crystal.

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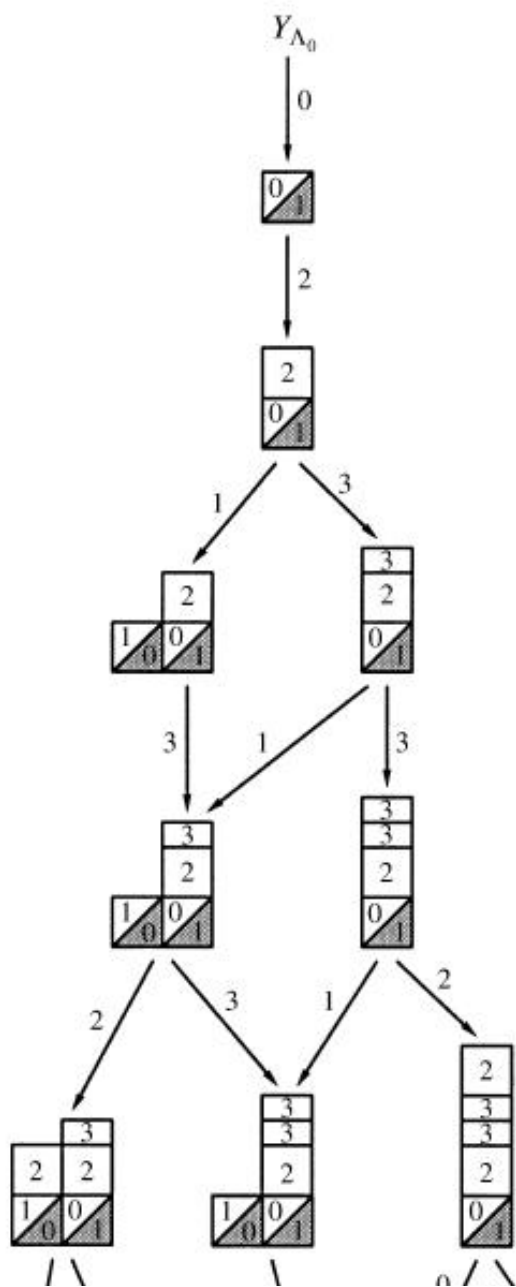
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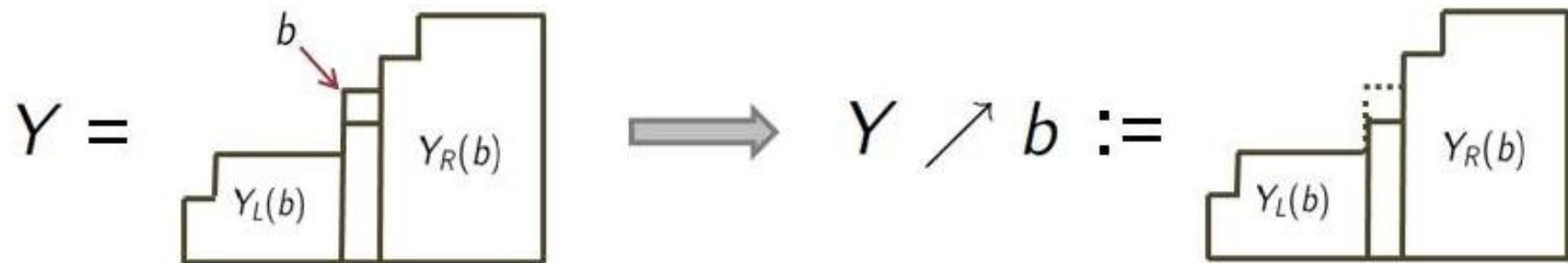
- ① $\mathcal{Z}(\lambda)$ is a $U_q(\mathfrak{g})$ -crystal.
- ② $\mathcal{Y}(\lambda) = \text{Conn}(\emptyset) \cong B(\lambda)$.

Example The crystal $\mathcal{Y}(\Lambda_0)$ for $B_3^{(1)}$

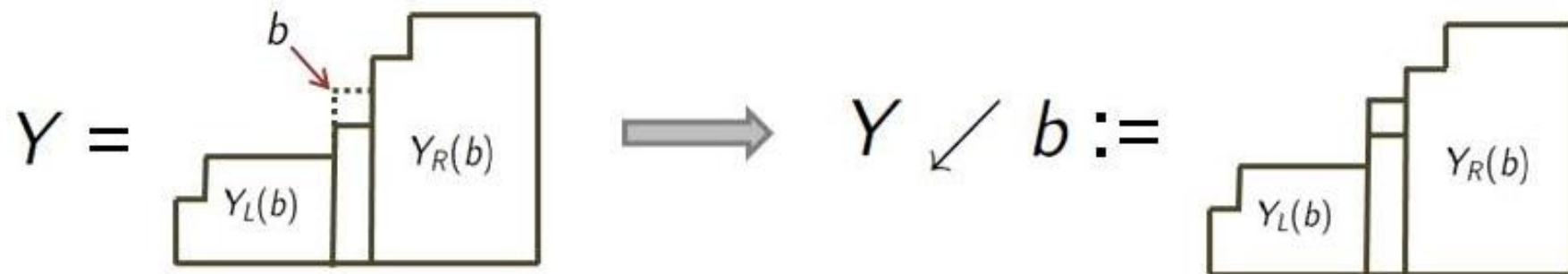


Define the actions of e_i, f_i ($i \in I$) on

$$\mathcal{F}(\lambda) = \bigoplus_{Z \in \mathcal{Z}(\lambda)} \mathbb{Q}(q)Z.$$


$$e_i Y = \sum_{b: \text{removable } i\text{-blocks}} q_i^{-R_i(b; Y)} (Y \nearrow b),$$

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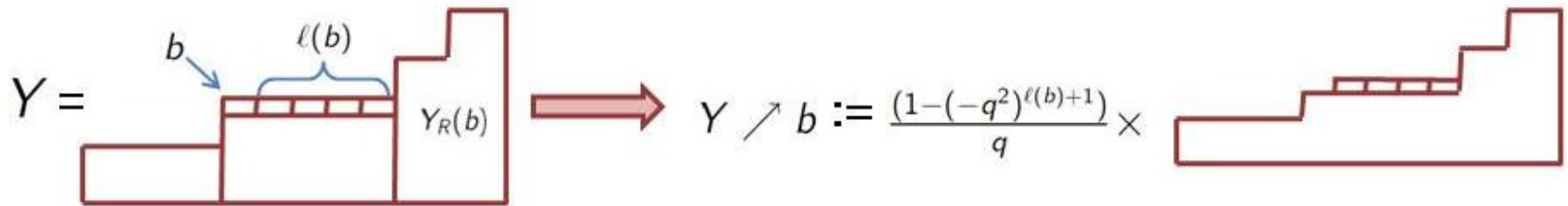
Type II  = 

Y : Young wall, b : removable i -block

If the i -signature of $b = --$ or $-$,

$\Rightarrow Y \nearrow b :=$ Young wall obtained by removing b from Y .

If the i -signature of $b = -+$, then



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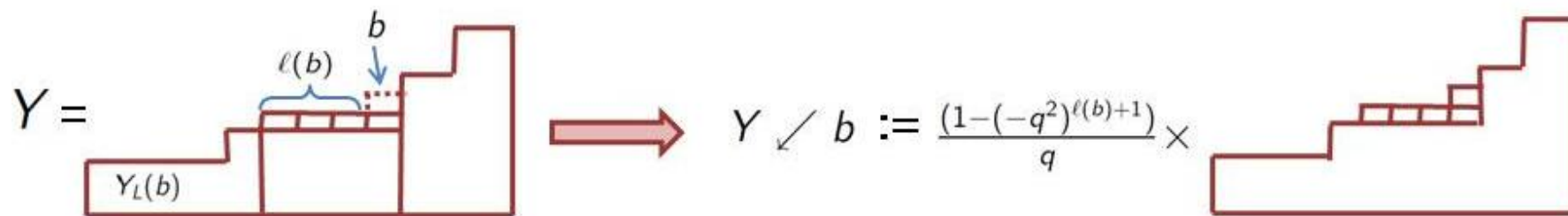
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Y : Young wall, b : admissible i -slot

If the i -signature of $b = ++$ or $+$,

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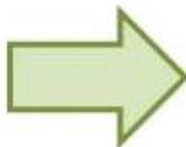
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Consider the following block, called a **virtually removable i -block**:



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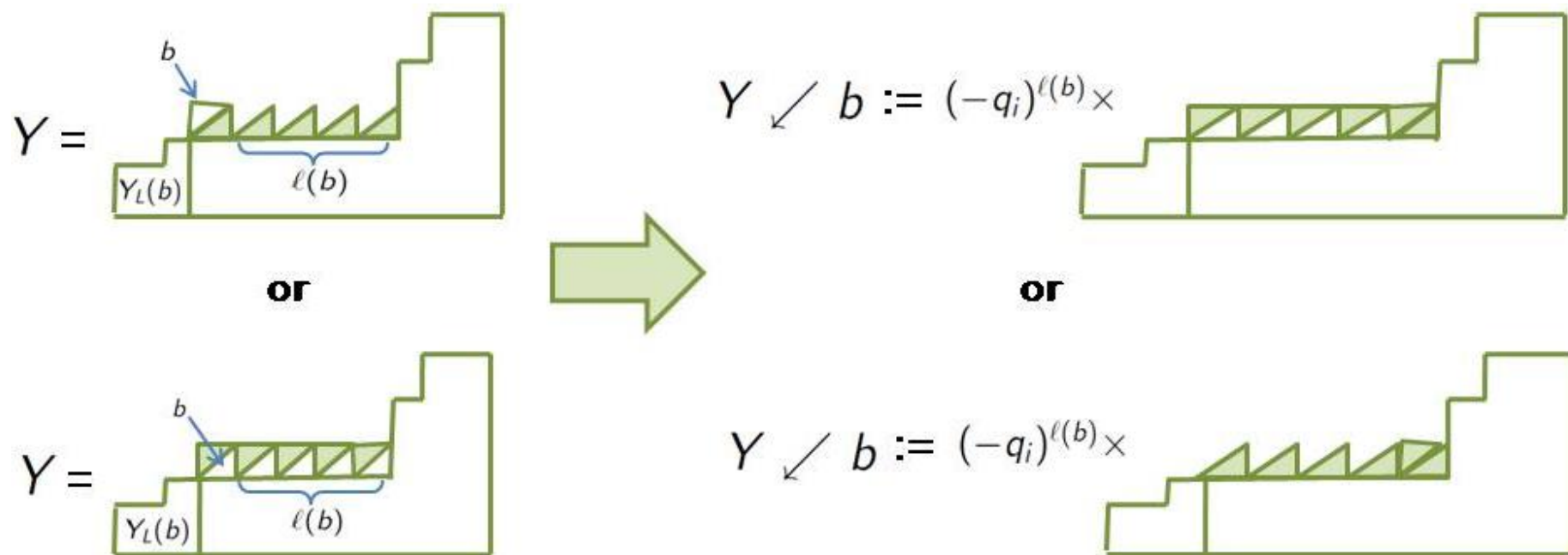
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Theorem(K., J.-H. Kwon 2008)

- 1 $\mathcal{F}(\lambda)$ is a $U_q(\mathfrak{g})$ -module in O_{int}^q .

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- ② Given $Y \in \mathcal{Y}(\lambda)$, we found an algorithm of computing

$$G(Y) = Y + \sum_{Y' \in \mathcal{Z}(\Lambda_0), Y' \triangleleft Y} K_{Y, Y'}(q) Y'.$$

$U_q(\mathfrak{g})$	$A = ?$
$\mathcal{Z}$: Young wall	?
$\mathcal{Y}$: reduced	?
$\mathcal{Y}(\lambda) \cong B(\lambda)$	?
$\mathcal{F}(\lambda)$, LLT	?

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s.t. i) $K_0(H) \cong U_{\mathbb{Z}}^-(\mathfrak{g})$, ii) $B(K_0(H)) \cong B(\infty)$,

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THANK YOU