1^{st} **PRIMA Congress**

Cloaking and Transformation Optics

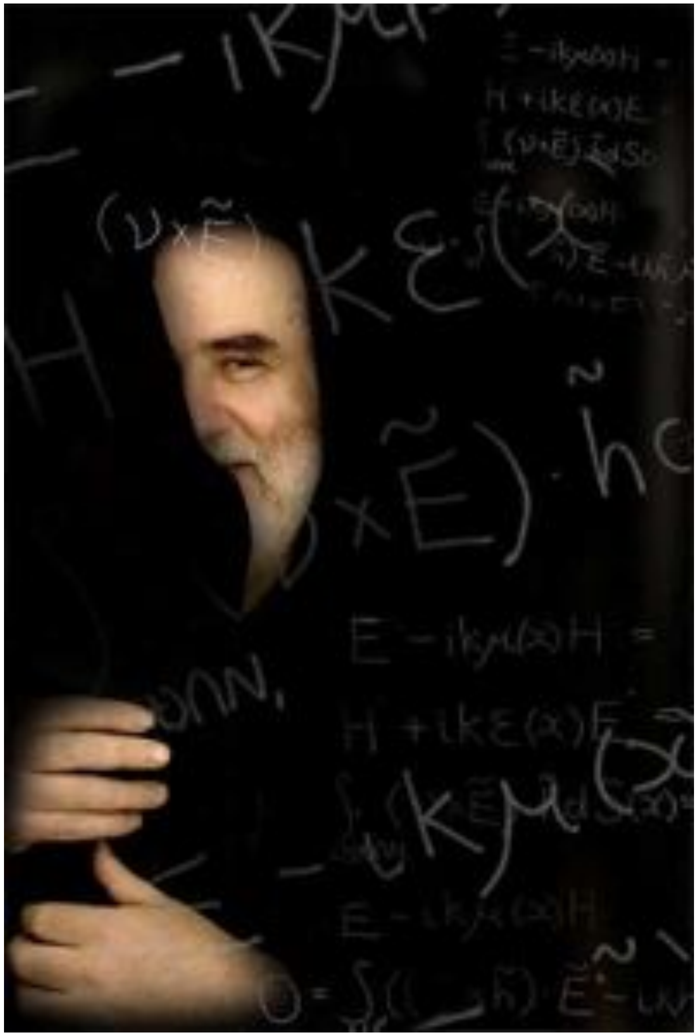
Gunther Uhlmann

University of Washington

Sydney, July 6, 2009

H. G. Wells: The Invisible Man (1897)

Susan Storm Richards: The Invisible Woman (1961)



Transformation Optics

Two Recent Articles in Science on Invisibility "Controlling Electromagnetic Fields", **J.B. Pendry, D. Schurig, D.R. Smith**, Science **312**, pp. 1780-1782, (June 2006).

Related article by **Ulf Leonhard** "Optical Conformal Mapping" in same issue.

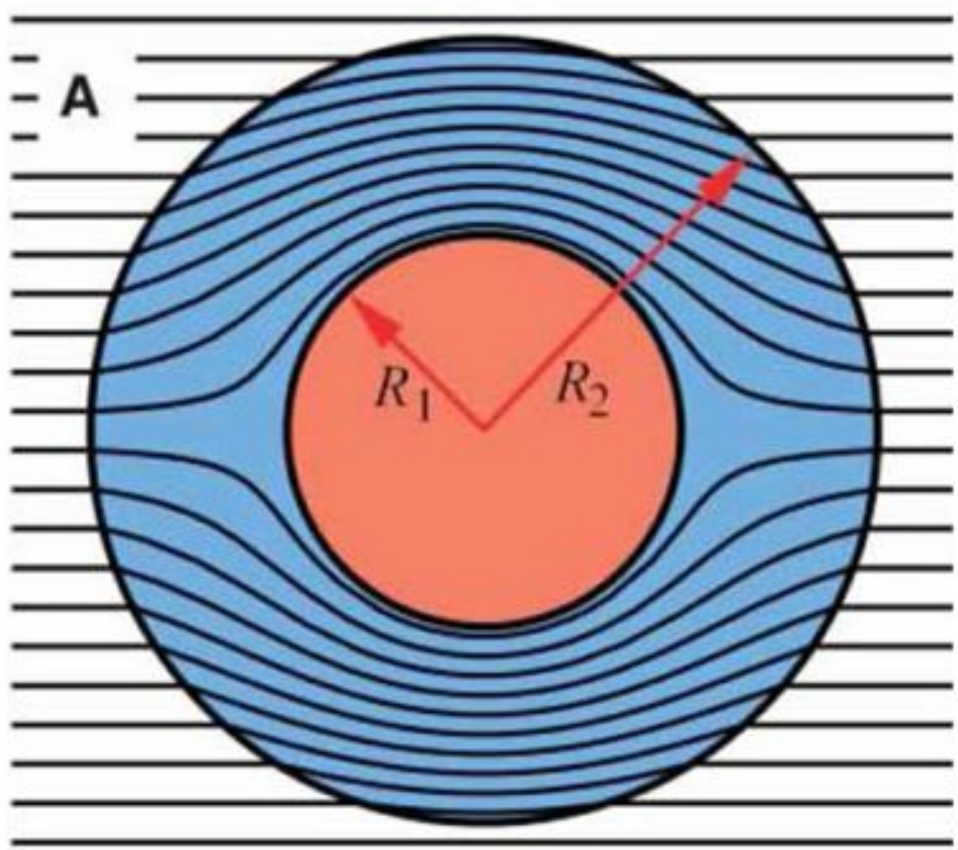
Earlier article of **A. Greenleaf, M. Lassas and G-U**, "Non-uniqueness for Calderón's problem", Math. Research Letters, 2003.

Transformation Optics

Science Magazine

No. 5, Breakthrough of 2006: THE ULTIMATE CAMOUFLAGE

"The real breakthrough may lie in the theoretical tools used to make the cloak. In such "transformation optics," researchers imagine–à la Einstein–warping empty space to bend the path of electromagnetic waves. A mathematical transformation then tells them how to mimic the bending by filling unwarped space with a material whose optical properties vary from point to point. The technique could be used to design antennas, shields, and myriad other devices. Any way you look at it, the ideas behind invisibility are likely to cast a long shadow."



From Pendry et al's paper

All Boundary measurements for the homogeneous conductivity $\gamma = 1$ and the degenerate conductivity $\tilde{\sigma}$ are the same

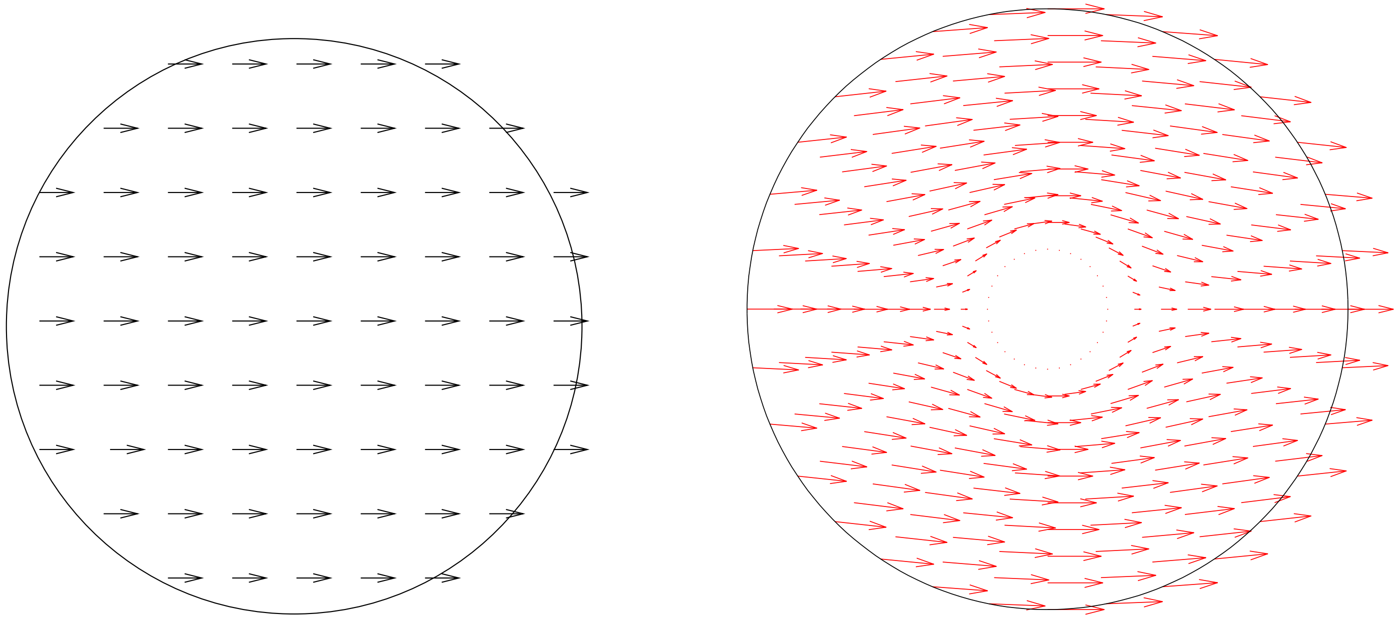
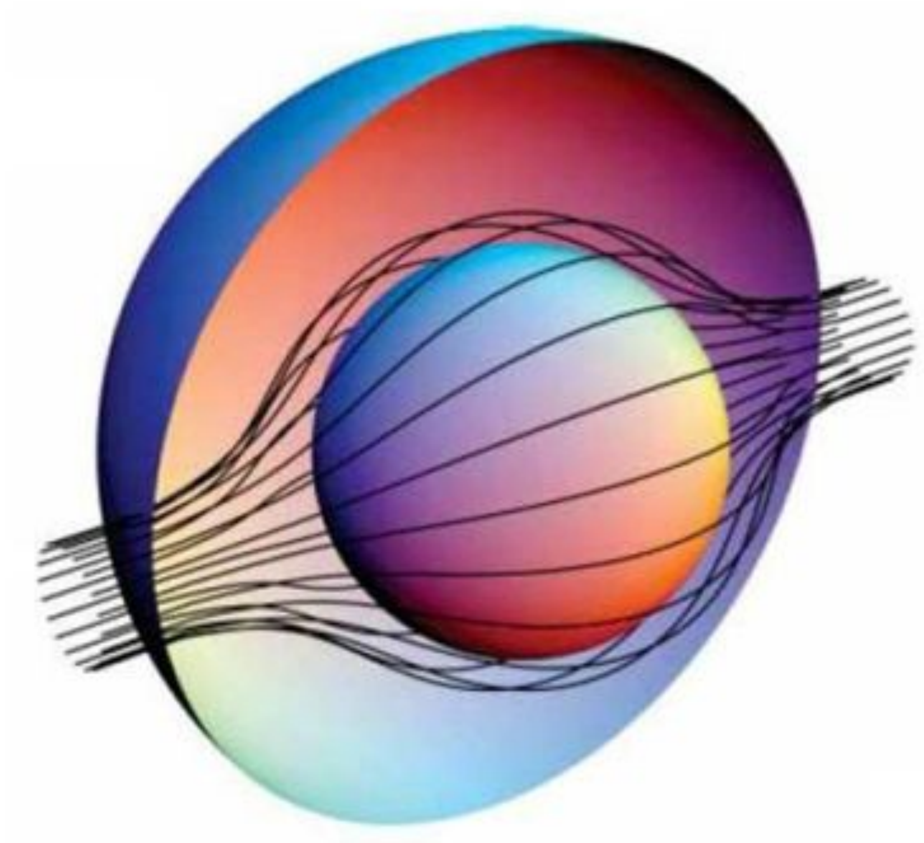


Figure: Analytic solutions for the currents

Based on work of Greenleaf-Lassas-U, MRL 2003

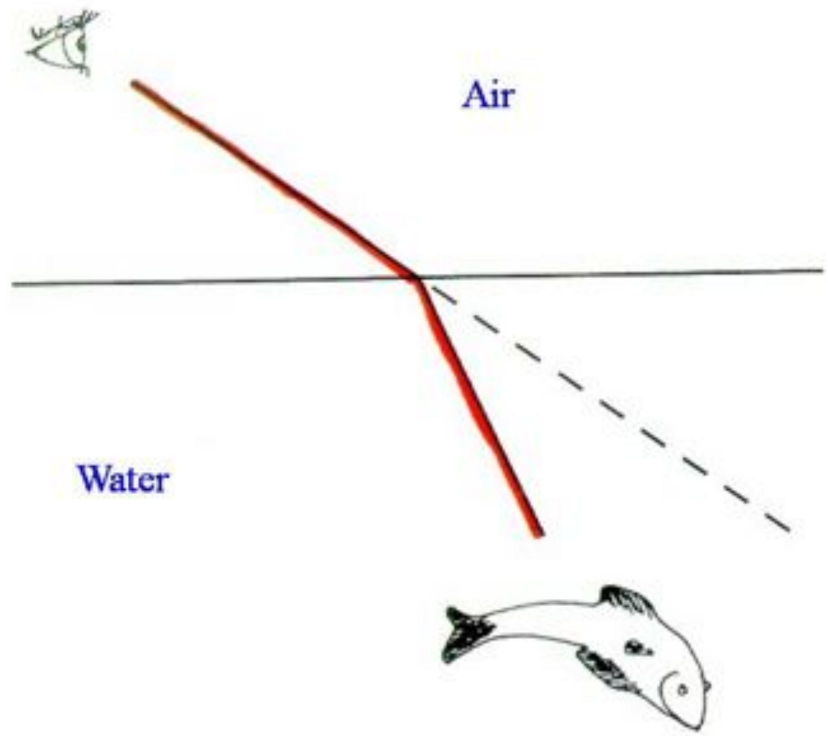
White Hole



From Pendry et al's paper

Index of Refraction

Fermat's Principle: Minimize Optical Length



Mirage

S. Cummer, B. Pope, D. Schurig, D. Smith and J. Pendry, “Full-wave simulations of electromagnetic cloaking structures”, Phys. Rev. E **74** 036621 (2006)

“It is open problem whether full-wave cloaking is possible, even in theory”

Answer: It is possible for all frequencies for electromagnetic waves. This is joint work with A. Greenleaf, M. Lassas and Y. Kurylev. (Comm. Math. Physics, 2007).

Model :Maxwell's Equations for Time Harmonic Waves

$$\nabla \times E - ik\mu(x)H = 0$$
$$\nabla \times H + ik\epsilon(x)E = 0$$
$$D = \epsilon E, B = \mu H$$
$$\text{div}D = 0, \text{div}B = 0$$

J. Maxwell

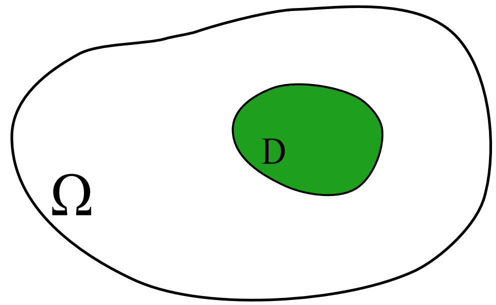
(E, H) Electromagnetic Field

D=electric displacement
B=magnetic displacement

$\epsilon(x)$=electric permittivity
$\mu(x)$=magnetic permeability

Index of refraction: $\sqrt{\epsilon\mu}$

What do we mean by invisibility?



Object to be cloaked in D. $(\epsilon(x), \mu(x))$ electromagnetic parameters of Ω, arbitrary on D.
We want to show that if Maxwell's equations are solved in Ω, boundary information of solutions on Ω is same as the case o

$$\epsilon(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mu(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(Homogeneous case)

We will consider a special case of anisotropic materials:

$\epsilon(x) = \mu(x) = g^{-1}\sqrt{\det g}$ were $g = (g_{ij})$ is a semipositive definite symmetric matrix

Similar (Helmholtz, Acoustic waves, Quantum waves)

$$\Delta_g E + k^2 E = 0$$
$$\Delta_g H + k^2 H = 0$$

$$\Delta_g = \text{Laplace-Beltrami operator}$$

$$= \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\; g^{ij} \frac{\partial}{\partial x_j}\right)$$

$(g^{ij}) = (g_{ij})^{-1}$.

(Helmholtz)

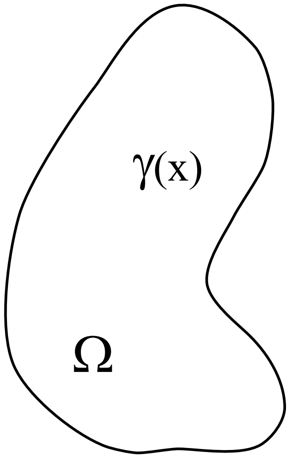
$$\Delta_g E + k^2 E = 0$$
$$\Delta_g H + k^2 H = 0$$

Consider first static case ($k = 0$)

$$\Delta_g E = \Delta_g H = 0$$

This problem in dimension $n \geq 3$ is equivalent to the Electrical Impedance Tomography (Calderón's Problem)

CALDERÓN'S PROBLEM



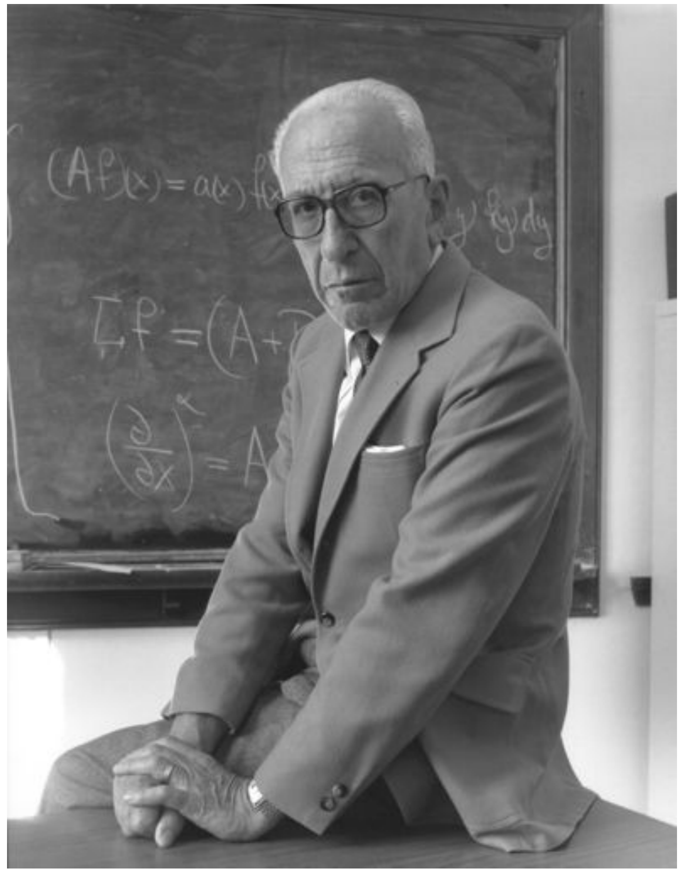
$\Omega \subset \mathbb{R}^n$

$(n = 2, 3)$

Can one determine the electrical conductivity of $\Omega, \gamma(x)$, by making voltage and current measurements at the boundary?
(Calderón; Geophysical prospection)

Early breast cancer detection

Normal breast tissue	0.3 mho
Cancerous breast tumor	2.0 mho



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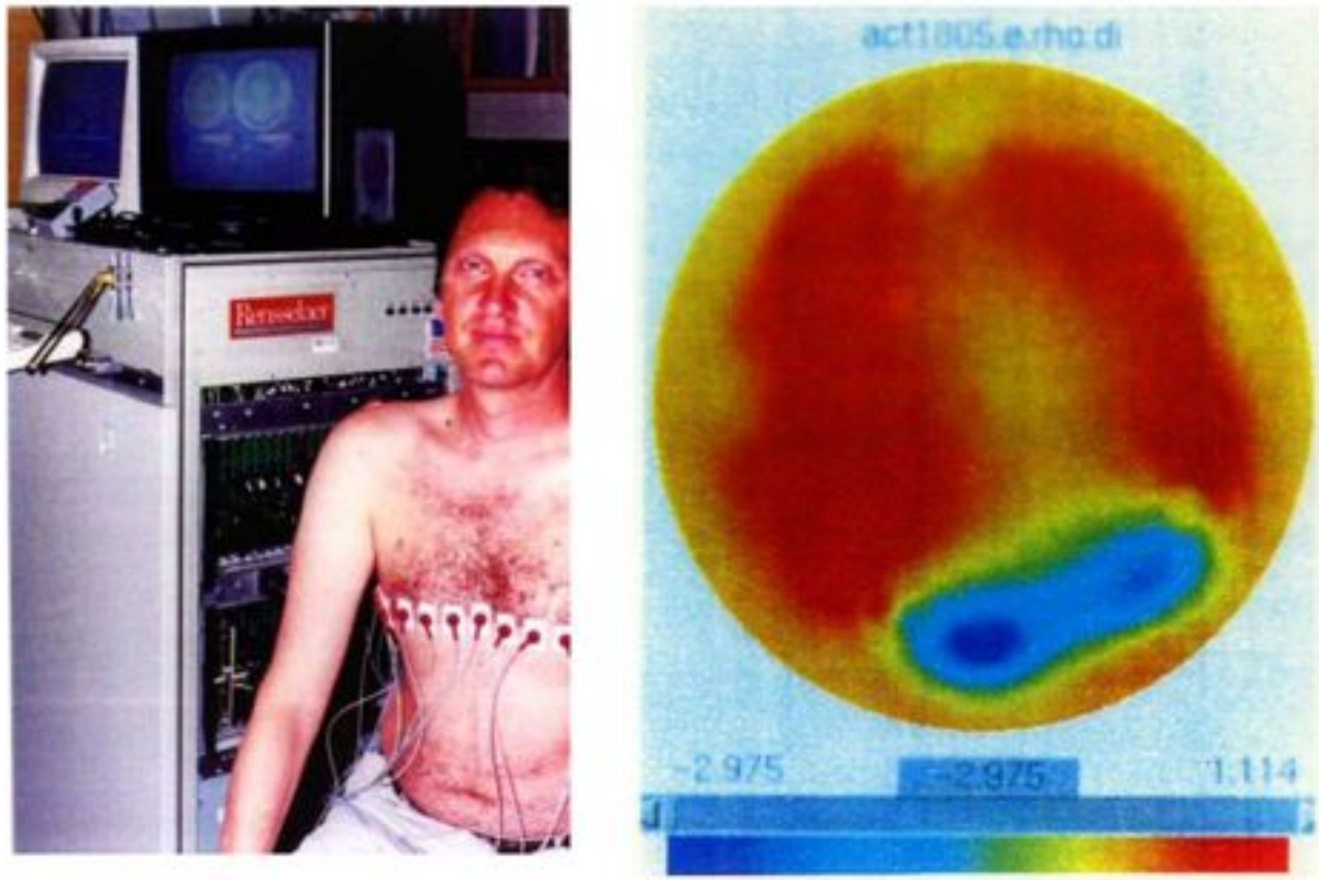
(Loading electro.mpg)



ACT3 imaging blood as it leaves the heart(blue) and fills the lungs (red) during systole.

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CALDERÓN'S PROBLEM (EIT)

Consider a body $\Omega \subset \mathbb{R}^n$. An electrical potential $u(x)$ causes the current

$$I(x) = \gamma(x)\nabla u$$

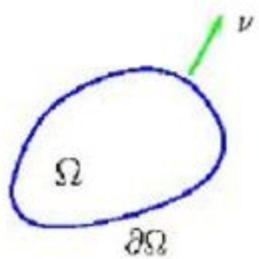
The conductivity $\gamma(x)$ can be isotropic, that is, scalar, or anisotropic, that is, a matrix valued function. If the current has no sources or sinks, we have

$$\text{div}(\gamma(x)\nabla u) = 0 \text{ in } \Omega$$

$$\boxed{\begin{aligned} \operatorname{div}(\gamma(x)\nabla u(x)) &= 0 \\ u\big|_{\partial\Omega} &= f \end{aligned}}$$

$\gamma(x)$ = conductivity,
f = voltage potential at $\partial\Omega$

Current flux at $\partial\Omega = (\nu \cdot \gamma \nabla u)\big|_{\partial\Omega}$ were ν is the unit outer normal.



Information is encoded in map

$$\boxed{\Lambda_\gamma(f) = \nu \cdot \gamma \nabla u\big|_{\partial\Omega}}$$

EIT (Calderón's inverse problem)

Does $\boxed{\Lambda_\gamma}$ determine $\boxed{\gamma}$?

Λ_γ = Dirichlet-to-Neumann map

Objects one cannot cloack

a) Isotropic case

$$0 < C_1 \leq \gamma(x) \leq C_2 \quad x \in \overline{\Omega}$$
$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2$$

- $n \geq 3$, Sylvester-U, 1987

Assumption, $\gamma \in C^2(\overline{\Omega})$ improved to $\gamma \in C^{3/2}(\overline{\Omega})$

Also valid for isotropic Helmoltz at fixed frequency $\Delta + k^2 n(x)$ $n \in L^\infty$ (more generally for the Schrödinger equation $\Delta + q), q \in L^\infty$.

- $n = 2$, $\gamma \in L^\infty$, (Astala-Päivärinta, 2006)

For isotropic Helmholtz equation $\Delta + k^2 n(x)$, $n(x) \in L^\infty$ and Schrödinger equation $\Delta + q$, $q \in L^\infty$ have recently been proven by Bukhgeim (2008).

Objects one cannot cloack

b) Anisotropic case

$\gamma = (\gamma^{ij})$ positive-definite in $\overline{\Omega}$, that is there exists $C_1, C_2 > 0$ such that

$$C_1|\xi|^2 \leq \sum \gamma^{ij}(x)\xi_i\xi_j \leq C_2|\xi|^2, x \in \overline{\Omega}$$

- $n = 3, \gamma^{ij}$ real-analytic (Lassas-U, 2001)

Also true for Helmholtz equation $\Delta_g + k^2$, g real analytic.

- $n = 2, \gamma^{ij} \in L^\infty$, (Astala-Lassas-Päivärinta, 2005)

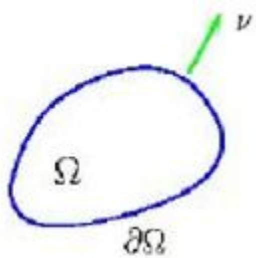
True for Helmholtz equation $\Delta_g + k^2$ (Bukhgeim, 2008)

DIRICHLET-TO-NEUMANN MAP

(M, g) compact Riemannian manifold with boundary.
Δ_g Laplace-Beltrami operator $g = (g_{ij})$ pos. def. symmetric matrix

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j}\right) \quad (g^{ij}) = (g_{ij})^{-1}$$

$$\Delta_g u = 0 \text{ on } M \qquad u\big|_{\partial M} = f$$



Conductivity:
$\gamma^{ij} = \sqrt{\det g}\, g^{ij}$

$$\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\Big|_{\partial M}$$

$\nu = (\nu^1, \cdots, \nu^n)$ unit-outer normal

$$\Delta_g u = 0$$
$$u\Big|_{\partial M} = f$$

$$\boxed{\Lambda_g(f) = \frac{\partial u}{\partial \nu_g} = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\Big|_{\partial M}}$$

current flux at ∂M

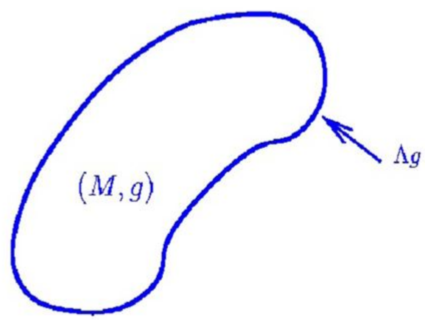
Inverse-problem (EIT)

Can we recover $\boxed{g}$ from $\boxed{\Lambda_g}$?

$\Lambda_g =$ Dirichlet-to-Neumann map or voltage to current map

ANOTHER MOTIVATION (STRING THEORY)

HOLOGRAPHY



Dirichlet-to-Neumann map is the "boundary-2pt function"

Inverse problem: Can we recover (M, g) (bulk) from boundary-2pt function ?

M. Parrati and R. Rabadan, Boundary rigidity and holography, JHEP 0401 (2004) 034

$$\Delta_g u = 0$$
$$u\Big|_{\partial M} = f$$

$$\Lambda_g(f) = \frac{\partial u}{\partial \nu_g}\Big|_{\partial M}$$

$$\Lambda_g \Rightarrow g \quad ?$$

Answer: No $\Lambda_{\psi^* g} = \Lambda_g$ where

$\psi : M \to M$ diffeomorphism, $\psi\Big|_{\partial M} =$ Identity and

$$\psi^* g = \left(D\psi \circ g \circ (D\psi)^T\right) \circ \psi$$

Theorem ($n \geq 3$) (Lassas-U, Lassas-Taylor-U) $(M, g_i), i = 1, 2$, real-analytic, connected, compact, Riemannian manifolds with boundary. Assume

$$\boxed{\Lambda_{g_1} = \Lambda_{g_2}}$$

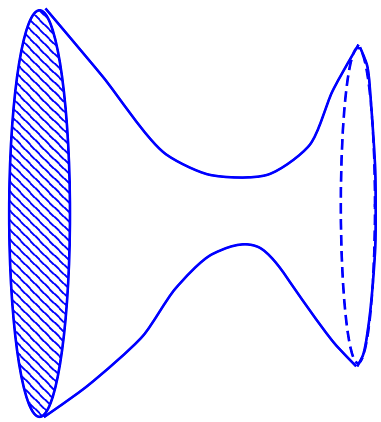
Then $\exists \psi : M \to M$ diffeomorphism, $\psi\Big|_{\partial M}$ = Identity, so that

$$\boxed{g_1 = \psi^* g_2}$$

One can also determine topology of M, as well (only need to know $\Lambda_g, \partial M$).

Non-uniqueness for EIT

Motivation (Greenleaf-Lassas-U, MRL, 2003)



When bridge connecting the two parts of the manifold gets narrower the boundary measurements give less information about isolated area.

When we realize the manifold in Euclidean space we should obtain conductivities whose boundary measurements give no information about certain parts of the domain.

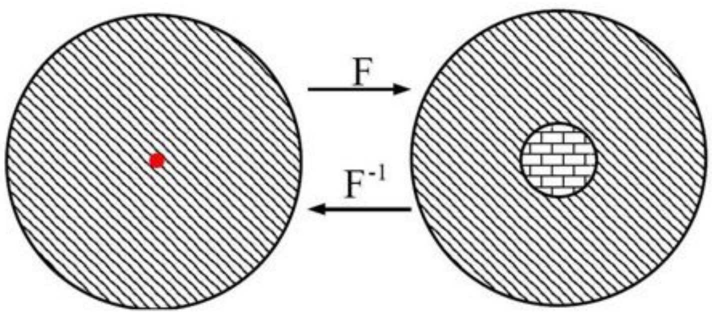
Greenleaf-Lassas-U (2003 MRL)

Let $\Omega = \mathcal{B}(0,2) \subset \mathbb{R}^3$, $D = \mathcal{B}(0,1)$ where $\mathcal{B}(0,r) = \{x \in \mathbb{R}^3; |x| < r\}$

$$F : \Omega \setminus \{0\} \to \Omega \setminus \overline{D}$$

$$\boxed{F(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}}$$

F diffeomorphism, $F\Big|_{\partial\Omega} = \text{Identity}$



$$\text{Let}\quad \begin{aligned} g &= \text{identity metric in } \mathcal{B}(0,2) \\ \widehat{g} &= (F^{-1})^* g \text{ on } \mathcal{B}(0,2)\setminus\mathcal{B}(0,1) \\ \widehat{\sigma} &= \text{conductivity associated to } \widehat{g} \end{aligned}$$

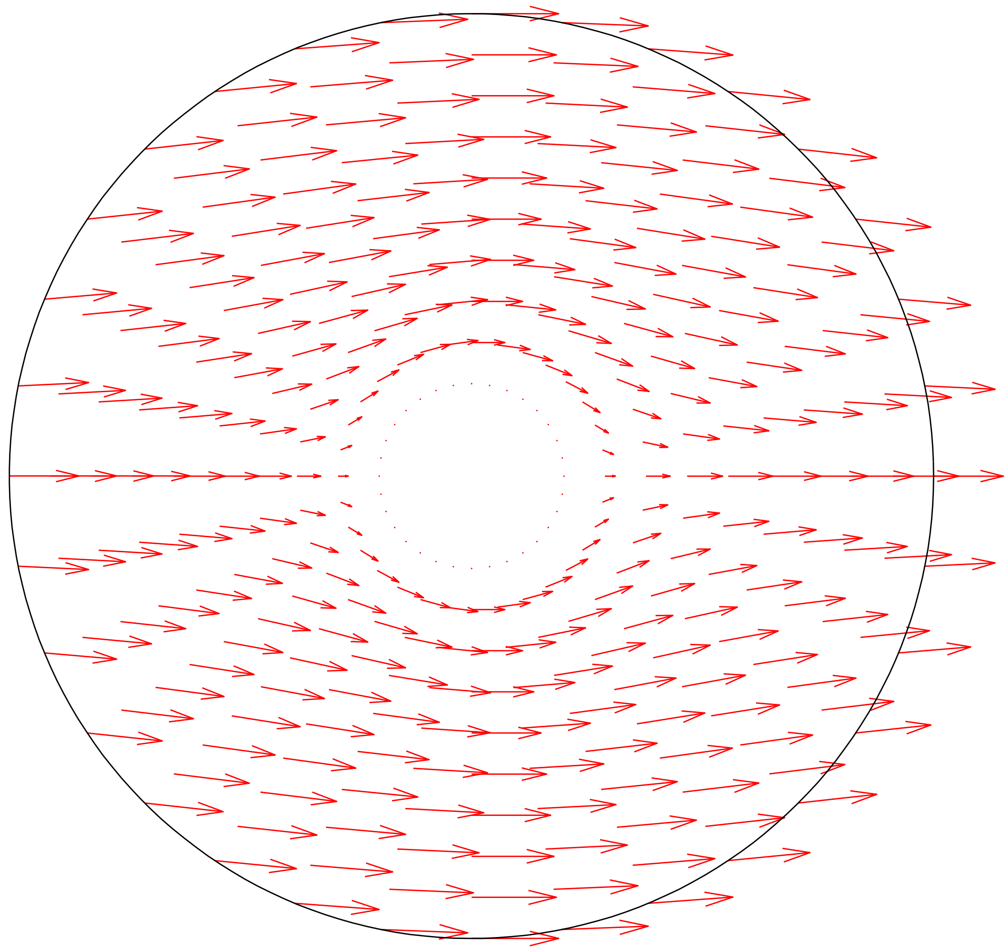
In spherical coordinates $(r,\phi,\theta) \to (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$

$$\widehat{\sigma} = \begin{pmatrix} 2(r-1)^2\sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}$$

Let $\widetilde{g}$ be the metric in $\mathcal{B}(0,2)$ (positive definite in $\mathcal{B}(0,1)$) s.t. $\widetilde{g} = \widehat{g}$ in $\mathcal{B}(0,2)\setminus\mathcal{B}(0,1)$. Then

Theorem (Greenleaf-Lassas-U 2003)

$$\boxed{\Lambda_{\widetilde{g}} = \Lambda_g}$$



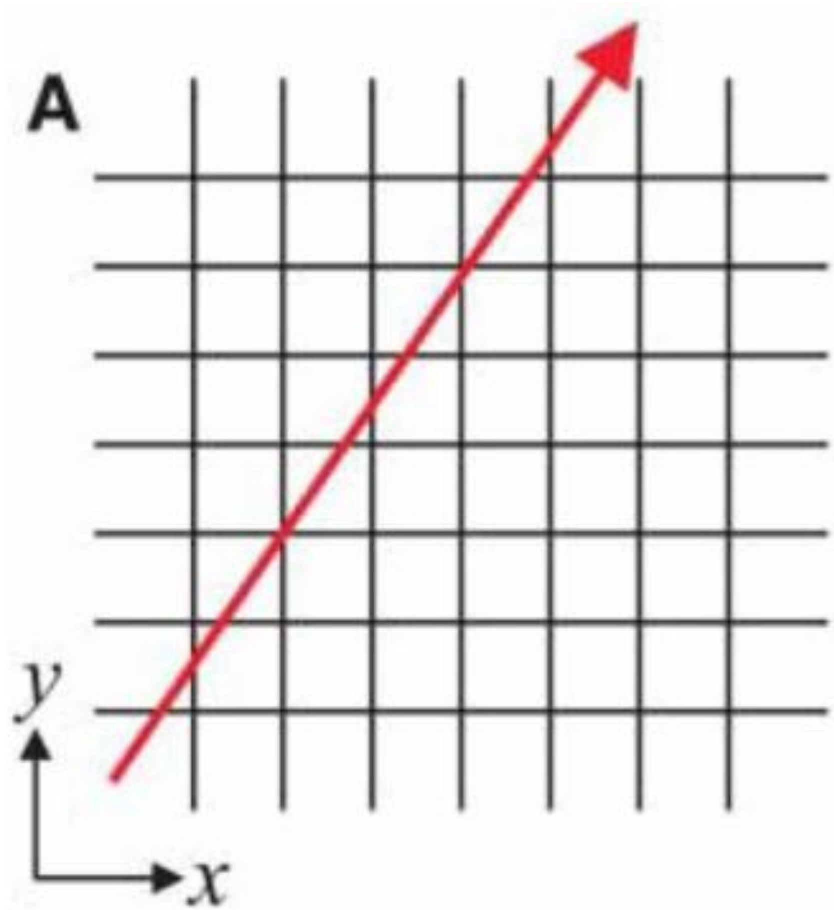
Based on work of Greenleaf-Lassas-U, MRL 2003

Pendry et al's construction (TRANSFORMATION MEDIA):

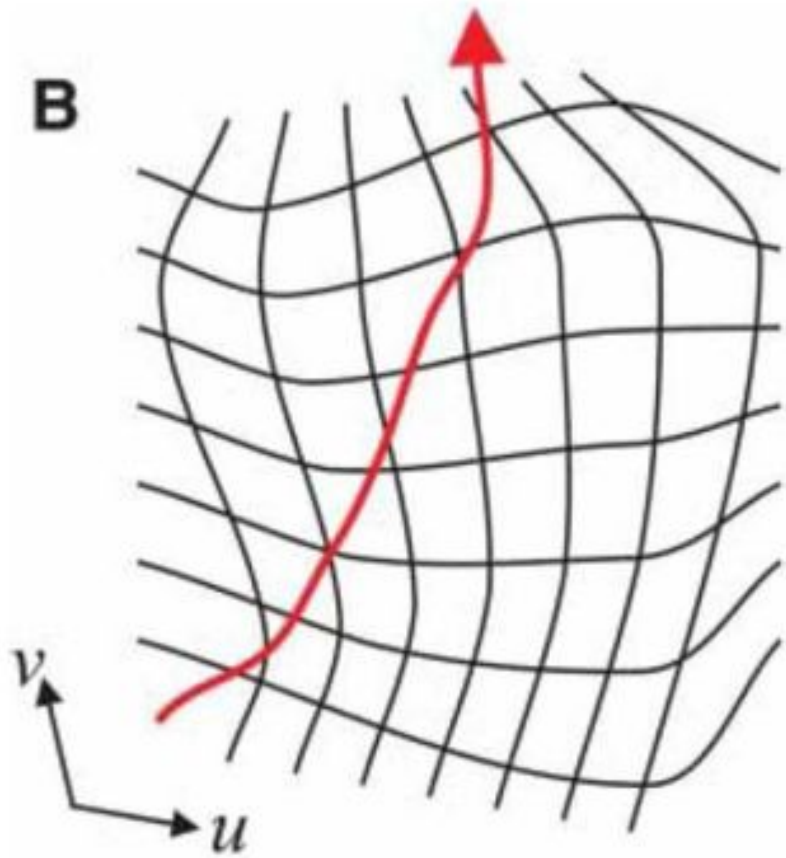
$\boxed{Step 1}$: Change coordinates and write Maxwell's equations in new coordinates.

In our notation: $(F^{-1})^* g$ where $g =$ Identity.

Recall that this preserves boundary measurements.



From Pendry et al's paper



From Pendry et al's paper

Pendry et al: Second Step

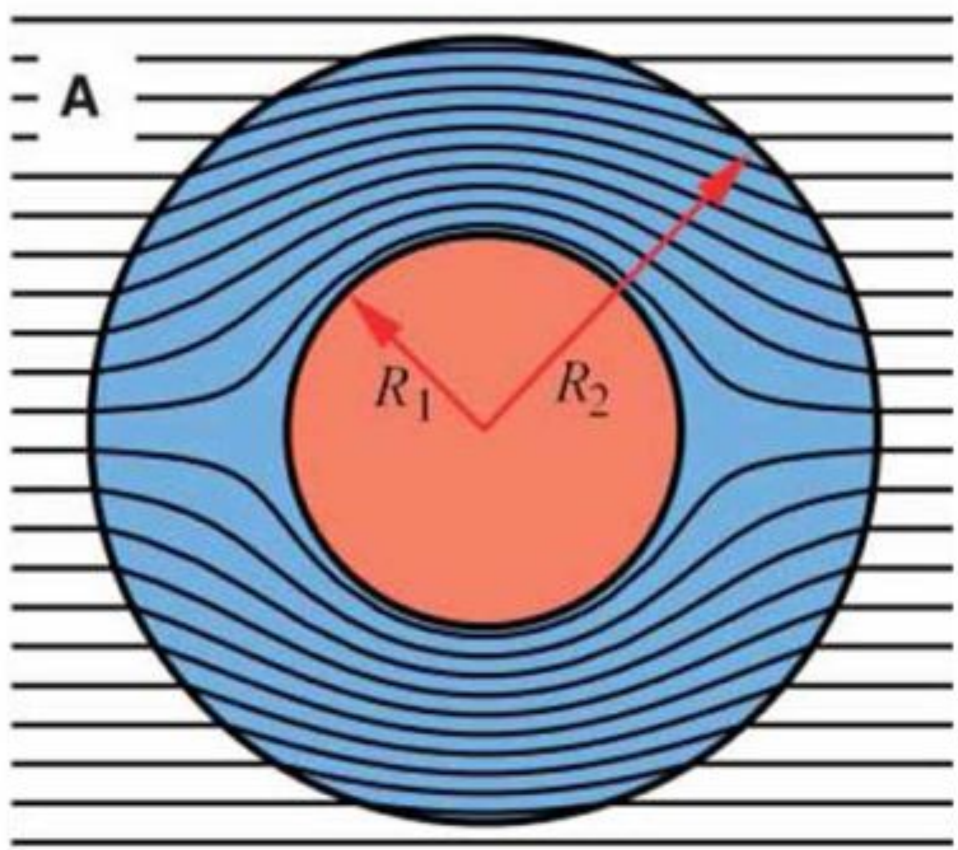
Spherical coordinates, (r, θ, ϕ). "All fields in the region $r < R_2$ are compressed into the region $R_1 < r < R_2$"

$$\begin{aligned} r' &= R_1 + \frac{r(R_2 - R_1)}{R_2} \\ \phi' &= \phi \\ \theta' &= \theta \end{aligned} \qquad \text{(F)}$$

Applying the coordinate transformation (F)

$$\begin{aligned} \epsilon'_{r'} &= \mu'_{r'} = \frac{R_2}{R_2 - R_1}\frac{(r' - R_1)^2}{r'^2} \\ \epsilon'_{\theta'} &= \mu'_{\theta'} = \frac{R_2}{R_2 - R_1} \\ \epsilon'_{\phi'} &= \mu'_{\phi'} = \frac{R_2}{R_2 - R_1} \end{aligned}$$

were ϵ is electric permittivity and μ is magnetic permeability



From Pendry et al's paper

Helmholtz Equation (Acoustic Cloaking)

$$\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j} \right) + k^2 u = 0 \quad (g^{ij}) = (g_{ij})^{-1}.$$

Acoustic equation with density $\rho = \sqrt{\det g} g^{ij}$ and bulk modulus $\lambda^{-1} = \sqrt{\det g}$

$$g^{-1} = \begin{pmatrix} 2(r-1)^2 \sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}$$

Theorem (Greenleaf-Kurylev-Lassas-U, CMP 2007)

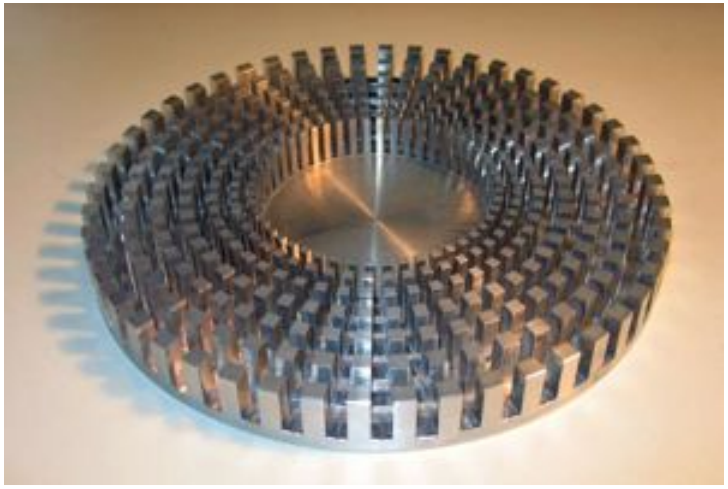
$$\Lambda_g = \Lambda_e$$

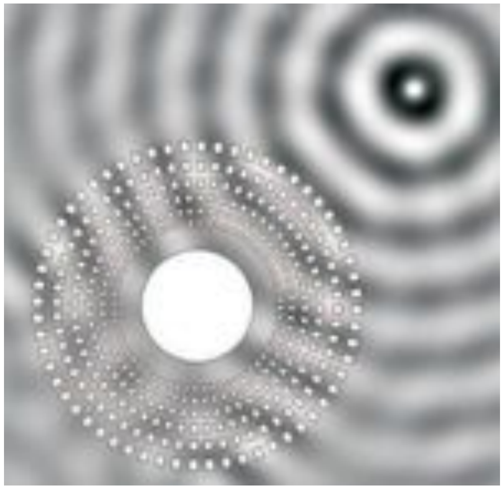
Acoustic Cloaking:

H. Chen and C. J. Chan, Appl. Phy. Lett., (2007)

S. Cummer et al, PRL (2008);

Tsunami Cloaking





"Broadband cylindrical acoustic cloak for linear surface waves in a fluid", M. Farhat et al, PRL (2008).

Isotropic Transformation Optics (Greenleaf-Kurylev-Lassas-U, PRL 2008, NJP 2008)

Acoustic and conductivity equation:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\gamma^{ij}\frac{\partial u}{\partial x_j}) + \frac{k^2}{\sqrt{\det\gamma}}u = 0.$$

$$F(x) = \begin{cases} x, & |x| > 2 \\ (1+\frac{|x|}{2})\frac{x}{|x|}, & 0 < |x| < 2 \end{cases}$$

$$\tilde{\gamma} = \begin{cases} F_*(\delta^{jk}), & x \in B(0,3) - B(0,1) \\ 2(\delta^{jk}), & x \in B(0,1) \end{cases}, \quad F_*\gamma = \frac{1}{|\det F|}(DF)\circ\gamma\circ(DF)^T\circ F^{-1}$$

$$(\tilde{\gamma}_R^{jk})(x) = \begin{cases} (\tilde{\gamma}^{jk})(x), & |x| > R \\ 2(\delta^{jk}), & |x| \le R \end{cases}, \quad 1 < R < 2,$$ non-singular anisotropic

($\widetilde{\gamma}_R^{jk}$ non-singular anisotropic)

Homogenization (Allaire, Cherkaev, Milton)

Construct isotropic conductivities $\widetilde{\gamma}_{R,\varepsilon}$ that give almost cloaking for acoustic and conductivity equation

$$\boxed{\Lambda_{\widetilde{\gamma}_{R,\varepsilon}} \xrightarrow[\varepsilon\to 0, R\to 1]{} \Lambda_e}, \qquad e = (\delta^{jk})$$

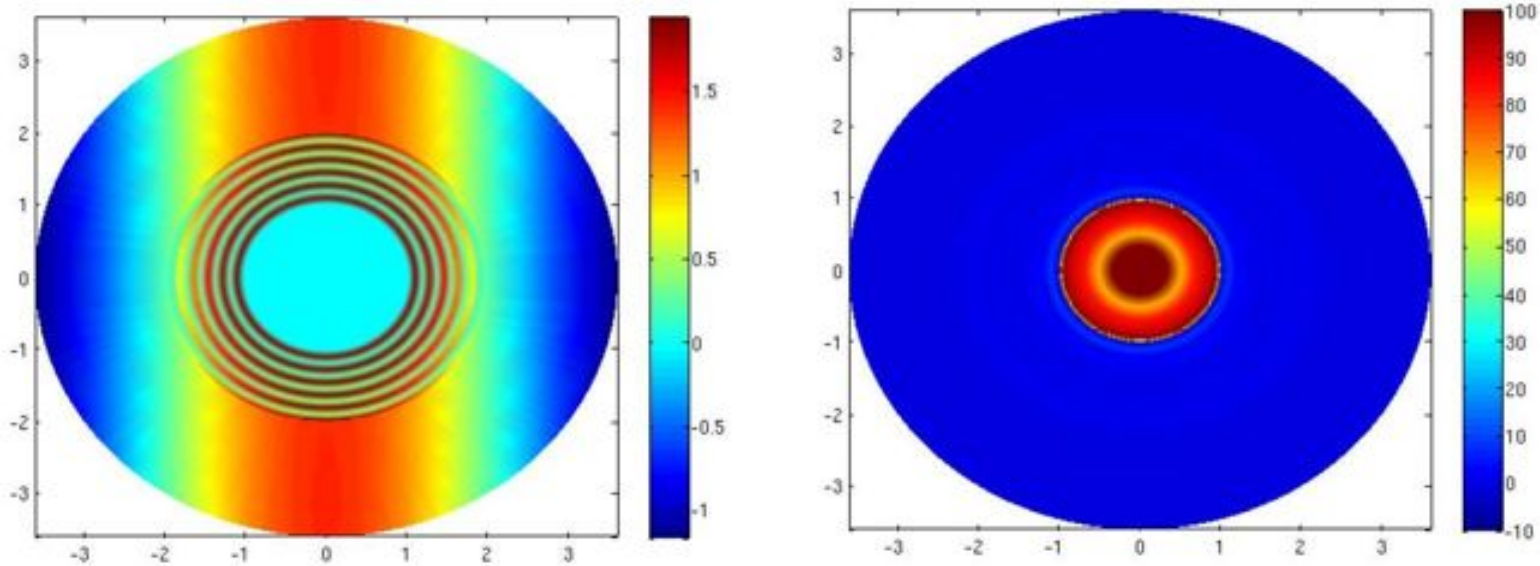
$$\boxed{\operatorname{div}(\widetilde{\gamma}_{R,\varepsilon}\nabla u) + \frac{k^2}{\sqrt{\det \widetilde{\gamma}_{R,\varepsilon}}} u = 0.}$$

Approximate quantum cloaks (Greenleaf-Kurylev-Lassas-U, PRL 2008, NJP 2008) E is not a Neumann eigenvalue in cloaked region

$$\varphi = \sqrt{\widetilde{\gamma}_{R,\varepsilon}} u,$$
$$(-\Delta + V_{E,R,\varepsilon})\varphi = E\varphi, \quad E = k$$
$$V_{E,R,\varepsilon} = \frac{\Delta\sqrt{\widetilde{\gamma}_{R,\varepsilon}}}{\sqrt{\widetilde{\gamma}_{R,\varepsilon}}} + E(1 - \frac{1}{\det \widetilde{\gamma}_{R,\varepsilon}})$$

W bounded potential on $B(0,1)$. Construct a sequence $V_n(E)$ s.t.

$$\boxed{\Lambda_{V_n(E)+W} \underset{n\to\infty}{\longrightarrow} \Lambda_0}$$



Left: Approximate cloak with E not a Neumann eigenvalue; matter wave passes unaltered.

Right: E a Neumann eigenvalue: cloak supports almost bound state.

Cloaking for Maxwell's equations (Passive Devices)

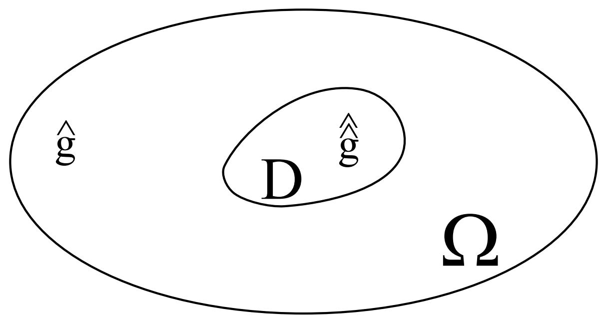
Model :Maxwell's Equations for Time Harmonic Waves

$$\nabla \times E - ik\mu(x)H = 0$$
$$\nabla \times H + ik\epsilon(x)E = 0$$
$$D = \epsilon E, B = \mu H$$
$$\text{div} D = 0, \text{div} B = 0$$

(E, H) Electromagnetic Field

D=electric displacement B=magnetic displacement	$\epsilon(x)$=electric permittivity $\mu(x)$=magnetic permeability

We will consider a special case of anisotropic materials:
$\epsilon(x) = \mu(x) = g^{-1}\sqrt{\det g}$ were $g = (g_{ij})$ is a semipositive definite symmetric matrix



$$\tilde{g} = \begin{cases} \hat{g} \text{ on } \Omega \setminus D, & \hat{g} = (F^{-1})^* e \\ \hat{\hat{g}} \text{ on } \overline{D}, & \hat{\hat{g}} \text{ Riemannian metric on } D \end{cases}$$

Theorem (Greenleaf-Lassas-Kurylev-U, CMP, 2007)

$$\Lambda_e = \Lambda_{\tilde{g}}$$

Here

$$\Lambda_g(E \wedge \nu) = H \wedge \nu$$

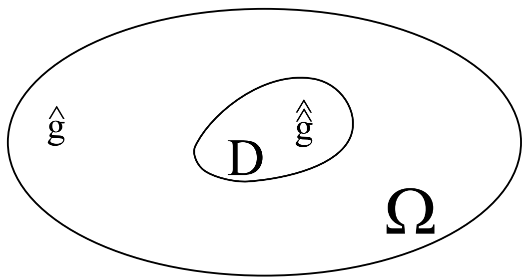
where ν is the inner-unit normal.
More generally we define the Cauchy data $(E \wedge \nu, H \wedge \nu)$.

Cloaking for Maxwell's equations (Active Devices)

Model :Maxwell's Equations for Time Harmonic Waves

$$\begin{aligned}
&\nabla \times E - ik\mu(x)H = 0 \\
&\nabla \times H + ik\epsilon(x)E = J \\
&D = \epsilon E, B = \mu H \\
&\mathrm{div} D = 0, \mathrm{div} B = 0
\end{aligned}$$

Where J is a non-zero current supported in cloaked region.



$$\widetilde{g} = \begin{cases} \widehat{g} \text{ on } \Omega \setminus D, & \widehat{g} = (F^{-1})^* e \\ \widehat{\widehat{g}} \text{ on } \overline{D}, & \widehat{\widehat{g}} \text{ singular on } \partial D \end{cases}$$

Theorem (Greenleaf-Kurylev-Lassas-U, CMP, 2007)

$$\Lambda_{\widetilde{g}} = \Lambda_e.$$

Based on work of A. Greenleaf, Y. Kurylev, M. Lassas–U

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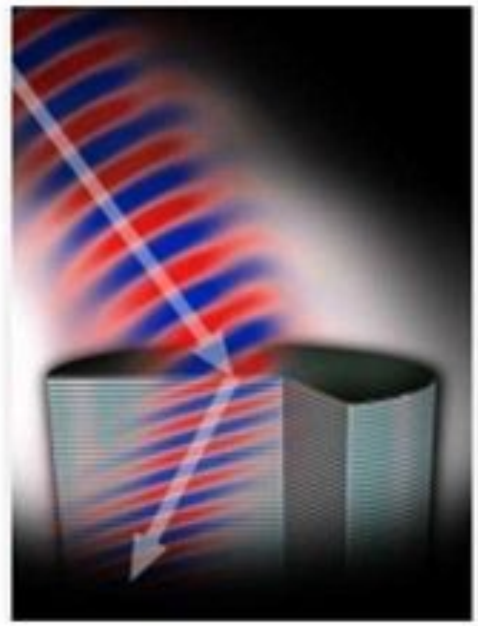
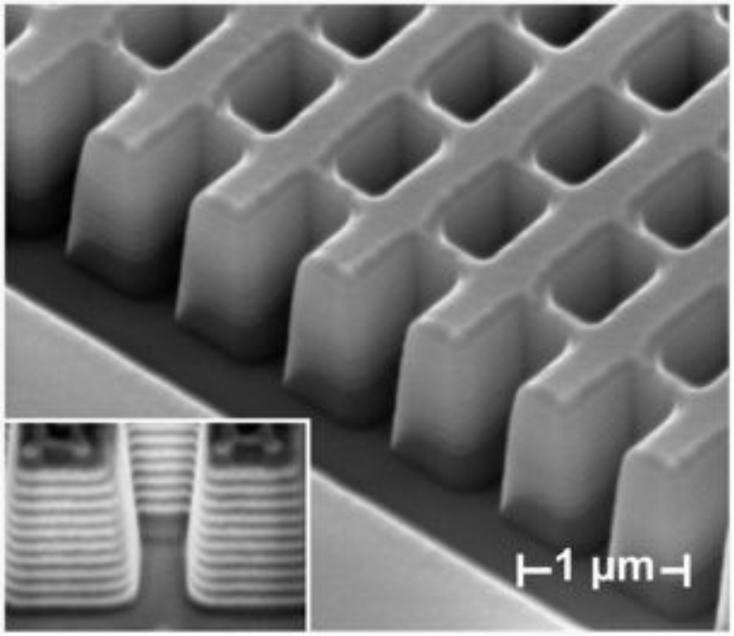
Metamaterials

$$\boxed{Negative\ Refractive\ Index : -\sqrt{\epsilon\mu}}$$



Credit: *Keith Drake*

Negative index of refraction for visible light

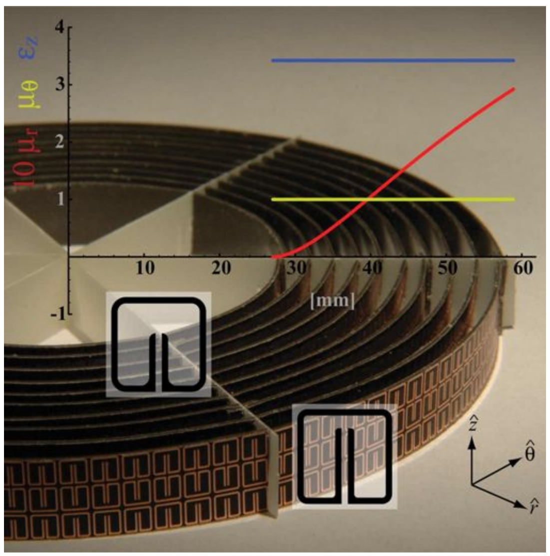


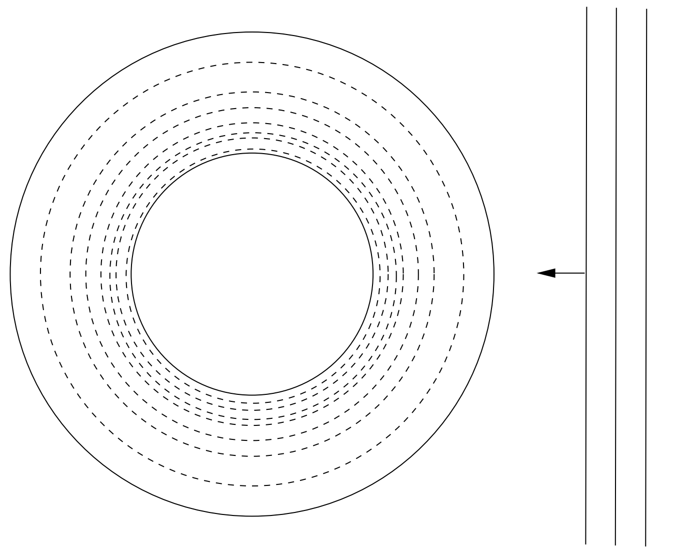
Nature 2008, Valentine et al, X. Zhang's group at UC Berkeley

Metamaterials

Cloaking

At microwave frequencies (D. Smith et al):





"Two-dimensional metamaterial structure exhibiting reduced visibility at 500nm", I.I. Smolyaninov et al, (Opt. Lett. 2008).

Electromagnetic Wormholes

Harry Potter's invisibility sleeve (Scientific American)

(A. Greenleaf, Y. Kurylev, M. Lassas–U, PRL, 2007)

How to construct a device that functions like a wormhole for electromagnetic fields?

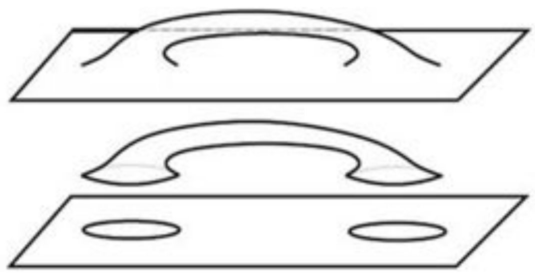
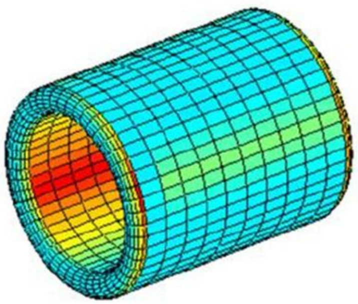


Figure: Schematic two-dimensional figure and an artistic interpretation of the wormhole device by Scientific American.

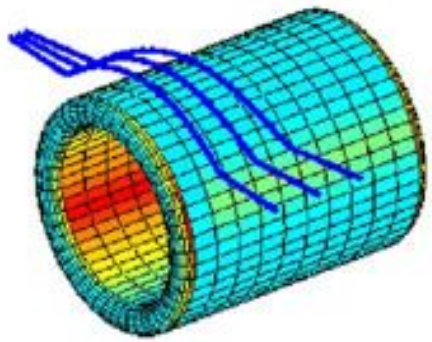
Blueprints of a Wormhole for Electromagnetic Waves

Take a cylinder

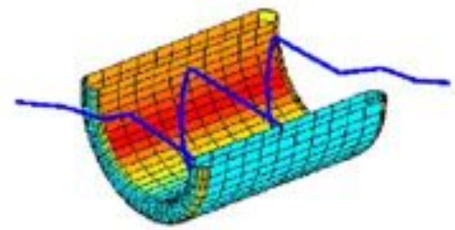


make parallel corrugation on its surface and coat it with suitable metamaterials. Such material has already been constructed for microwaves. Studies on optical frequencies are going on.

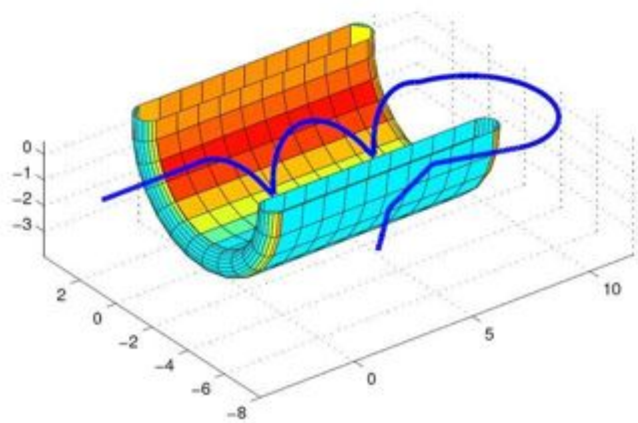
Ray tracing simulations:



Rays travelling outside.

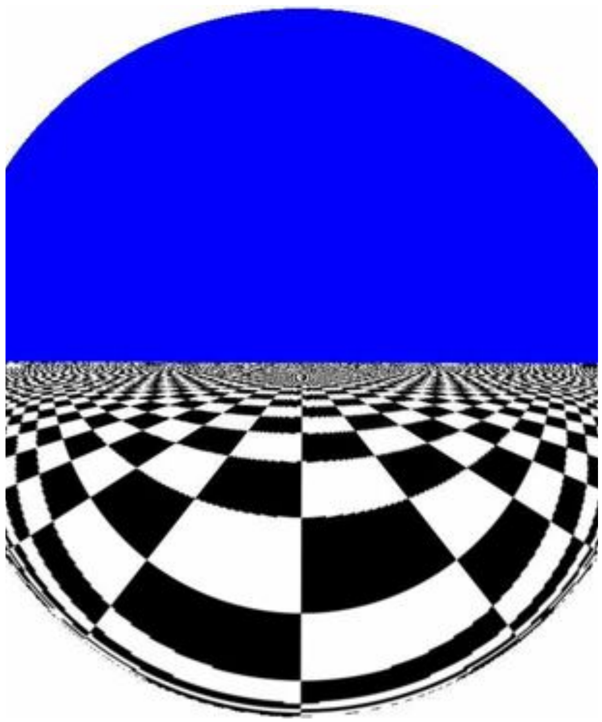


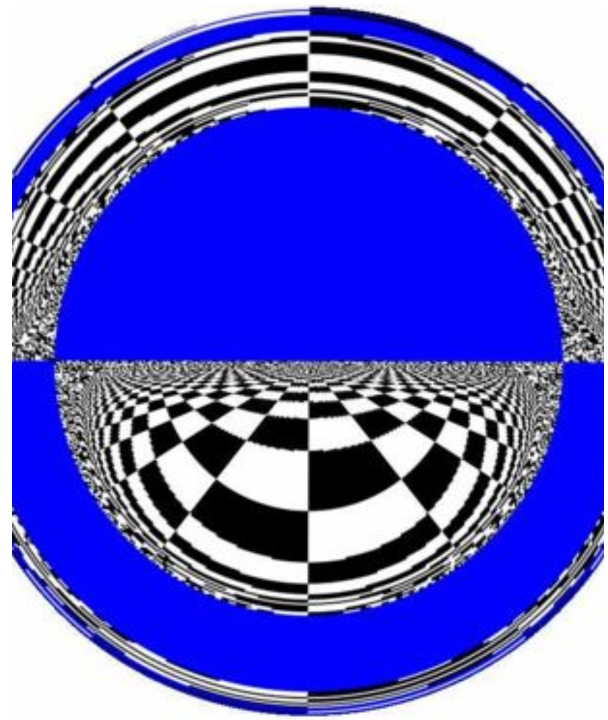
A ray traveling inside.



A ray coming back

Ray tracing simulations: the end of a wormhole





Length of handle $<< 1$. Length of handle ≈ 1.

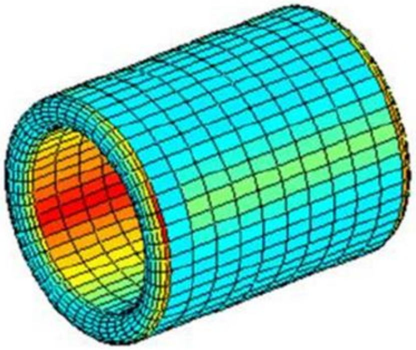
An end of the wormhole is sphere. The other end is over an infinite chess board and under the blue sky.

WHAT TO DO WITH A WORMHOLE

- OPTICAL CABLES: the wormhole could act as a perfect optical fibre that is invisible from the outside — but only at a single wavelength, the one the tube is designed to carry.
- OPTICAL COMPUTERS: data-processing elements for conducting logical operations on light signals could be placed inside the wormhole, so that all one would see of the computer is input and output sockets.
- THREE-DIMENSIONAL (3D) VIDEO DISPLAYS: each 3D pixel (voxel) of a cube-shaped space could be wired up with a separate wormhole, so that light fed in at the other end would appear in the voxel, creating a 3D image with invisible wiring.
- MAGNETIC MONOPOLES: the existence of magnetic particles that have only a 'north' or a 'south' pole has been long debated by physicists. But an artificial monopole could be made by conducting the magnetic field lines of just one pole of a magnet into a wormhole, so that the other end would act like a monopole.
- MAGNETIC RESONANCE IMAGING (MRI): a wormhole could transport the particles used for MRI, such as magnetic nanoparticles, to the imaged area without disturbing the applied magnetic field, which would allow high-quality images to be obtained. P.B.

Nature News (Nov. 15, 2007)

Possible applications in future:



- Invisible optical cables.
- Components for optical computers.
- 3D video displays: ends of invisible tunnels work as light source in 3D voxels.
- Light beam collimation.
- Virtual magnetic monopoles.
- Scopes for Magnetic Resonance Imaging devices.

Optics: Watch your back

(Kosmas L. Tsakmakidis and Ortwin Hess)
Nature 451, 27(January 3, 2008)