Categorical crepant resolutions of higher dimensional simple singularities

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  triangulated category: shift functor $[1]$, *distinguished triangles* $a \to b \to c \to a[1]$ instead of exact sequences $0 \to a \to b \to c \to 0$.

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- different (but related) varieties may have equivalent derived categories. (more symmetries)
- $\mathrm{Coh}(X)$ is a *heart* of $D^b(\mathrm{Coh}(X))$ w.r.t. t-structure. There are many hearts in a triangulated category.
- Example: $D^b(\mathrm{Coh}(\mathbf{P}^n)) \cong D^b(\text{Mod-}R)$ for a finite dimensional non-commutative k-algebra R (representations of a quiver algebra).

Derived category of coherent sheaves (continued)

1. *finite type*: $\dim_k \sum_{p \in \mathbf{Z}} \mathrm{Hom}^p(a, b) < \infty$.
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Example: If X is n-dim Calabi-Yau variety, $S \cong [n]$ (*n-Calabi-Yau category*)

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- *minimal model*: no more (-1)-curve

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- $\forall a \in D^b(\mathrm{Coh}(X))$, $b \to a \to c \to b[1]$. $c \in \langle c_0 \rangle$, $b = f^* f_* a \in D^b(\mathrm{Coh}(X'))$.

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- Remark: $D^b(\mathrm{Coh}(X))$ has no *orthogonal* decomposition.
- Corollary: If n-Calabi-Yau category, no SOD.
- Proof: If $\mathcal{B} \perp \mathcal{C}$, then $\mathcal{C} \perp \mathcal{B}$.

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One of the following output:

1. (MM): K_{X_m} is nef, $(K_{X_m} \cdot C) \geq 0, \forall C$. (*minimal model*)
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- Example: If $h : X \to S$ arbitrary resolution of singularities, a relative minimal model $h_m : X_m \to S$ is a *minimal resolution*.

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- If $n = n'$, $f : X \dashrightarrow X'$ is a *flop*.
- $\mu_*(\mu')^* : D^b(\mathrm{Coh}(X)) \cong D^b(\mathrm{Coh}(X'))$.

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- In particular, if $\mu^* K_X = (\mu')^* K_{X'}$, then $D^b(\mathrm{Coh}(X)) \cong D^b(\mathrm{Coh}(X'))$.

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- McKay correspondence: $D^b(\mathrm{Coh}(\mathcal{X})) \cong D^b(\mathrm{Coh}(Y))$.

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- We look for *categorical crepant resolution* by taking *categorical minimal resolutions*.

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- Both are relatively 2-Calabi-Yau categories: $S_{\mathcal{D}(X)}(c) = c[2]$, and $S_{\mathcal{D}(X)}(c^{\pm}) = c^{\pm}[2]$.

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- $S^3_{\mathcal{D}(X)} \cong [7]$. (relatively 7/3-Calabi-Yau category)

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- There are sheaves c_1, c_2, c_3 supported on $E_1 \cup E_2$.
  $0 \to c_1 \to c_2 \to c_3 \to 0$.
- $\{a \in \mathcal{D}_X \mid f_* a = 0\} = \langle c_1, c_2 \rangle$.
- $S_{\mathcal{D}(X)}(c_1) = c_2[2]$, $S_{\mathcal{D}(X)}(c_2) = c_3[2]$, $S_{\mathcal{D}(X)}(c_3) = c_1[3]$.
- $S^3_{\mathcal{D}(X)} \cong [7]$. (relatively 7/3-Calabi-Yau category)
- *Question: Let X be a variety with canonical singularities. Then does there exist a categorical minimal resolution whose relative part has a fractionally crepant filtation?*