RANDOM WALK IN RANDOM SCENERY

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§ INTRODUCTION

Main question:

> What do you see when you randomly move through a random landscape?

In this talk we consider a random walk on a randomly colored lattice and ask what are the properties of the sequence of colors encountered by the walk.

Two main ingredients:

- Random Walk: $X = \{X_n\}_{n \in \mathbb{Z}}$

i.i.d. steps drawn from $\mathbb{Z}^d$ according to any distribution m, and

$$S_0 = 0, \quad S_n - S_{n-1} = X_n \ (n \in \mathbb{Z}).$$

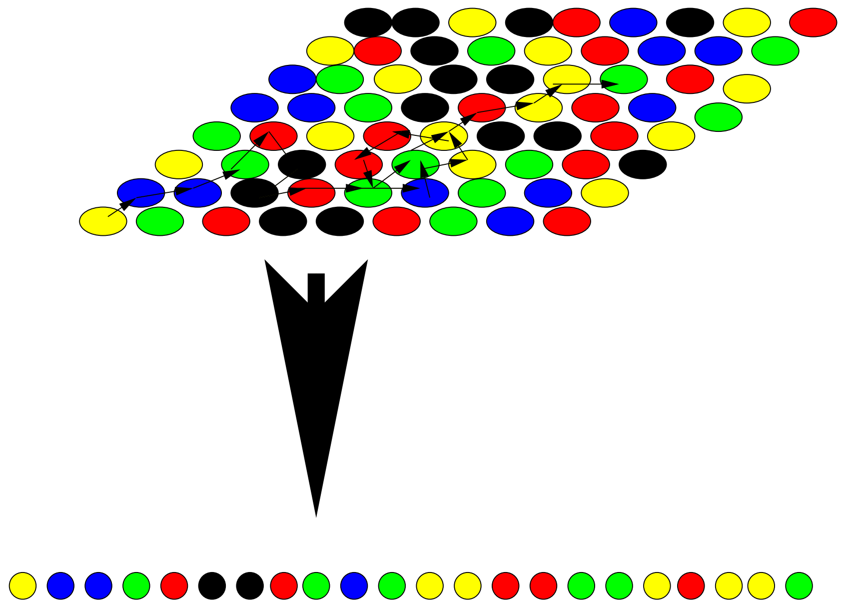
- Random Scenery: $C = \{C_z\}_{z \in \mathbb{Z}^d}$

i.i.d. colors drawn from a finite set F uniformly.

The composition

$$Z = \{Z_n\}_{n\in\mathbb{Z}} \quad \text{with} \quad Z_n = (C_{S_n})$$

is called the random color record.



The goal of this talk is to look at Z from three different angles:

(1) Ergodic properties.

(2) Scenery recovery.

(3) Bad configurations.

§ Ergodic properties

Understanding the ergodic properties of Z amounts to answering the following question:

Given that the color record is known in the full past, what can be said about the color record in the future?

BASIC FACT: (Meilijson 1974) Z *is stationary and has a Trivial Right Tail.*

We consider two further properties:

- Bernoulli
  = isomorphic to an i.i.d. process.

- Weak Bernoulli
  = past and far future are asymptotically independent.

Ordering:

$$\boxed{\text{WB} \subsetneq \text{B} \subsetneq \text{TRT}}$$

THEOREM: (dH & Steif 1997) Z *is* B *if and only if the random walk is transient.*

THEOREM: (dH & Steif 1997) Z *is WB if and only if the random walk is strongly transient.*

The proofs are based on coupling representations of B and WB. The "only if" part of the first theorem is subject to scenery recovery and to some mild regularity assumptions on m.

Typical examples:

- $d \geq 1$, $m = \frac{1}{2d} \sum_{e:|e|=1} \delta_e$.

not B	$d = 1, 2,$
B, not WB	$d = 3, 4,$
WB	$d \geq 5.$

- $d = 1$, $m = \frac{1}{\mathcal{N}} \sum_{x \neq 0} |x|^{-\alpha} \delta_x$.

not B	$\alpha \geq 2,$
B, not WB	$\frac{3}{2} \leq \alpha < 2,$
WB	$1 < \alpha < \frac{3}{2}.$

§ SCENERY RECOVERY

Scenery recovery is the inverse problem of recovering C from Z:

> By observing a single realization of the forward color record, can the full color scenery be recovered without knowing the walk?

Remarkably, the answer is sometimes yes.

A priori, scenery recovery is possible only for recurrent random walk, modulo symmetries of the random walk (like translations and reflections), and with probability 1.

THEOREM: (Löwe, Matzinger & Merkl 2004) *In* $d = 1$, *scenery recovery is possible when the number of available jumps is strictly less than the number of available colors.*

THEOREM: (Löwe & Matzinger 2002) *In* $d = 2$, *scenery recovery is possible for simple random walk when the number of available colors is very large.*

THEOREM: (Matzinger & Rolles 2003) *In* $d = 1$, *scenery recovery can be done in polynomial time and is stable against small random errors.*

§ BAD CONFIGURATIONS

A configuration η of the color record is called bad if there is an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there is some configuration ζ such that

$$\Big\| \mathbb{P}\Big(Z_0 \in \cdot \mid Z_{\mathbb{Z}\setminus 0} = \eta_{\mathbb{Z}\setminus 0}\Big) - \mathbb{P}\Big(Z_0 \in \cdot \mid Z_{\mathbb{Z}\setminus 0} = \eta_{[-N,N]\setminus 0} \vee \zeta_{\mathbb{Z}\setminus[-N,N]}\Big) \Big\|_{tv} \geq \epsilon.$$

The bad configurations are the discontinuity points of the conditional probabilities at the origin.

CONJECTURE: (dH & Steif, work in progress)

Consider the case

$$d = 1, \qquad m(1) = p, \quad m(-1) = 1 - p, \qquad F = \{B, W\}.$$

Then

(a) $\frac{1}{2} \leq p < \frac{4}{5}$*: All configurations are bad.*

(b) $\frac{4}{5} < p \leq 1$*: All configurations are good.*

This is a remarkable dichotomy.

The underlying reason is that an all black color record in a huge time interval outside a finite time interval may bias the color of the origin to be black, no matter what the color record inside the finite interval is.

So far, most results for RWRS are restricted to i.i.d. colors. Recently, some progress has been made for dependent colors, e.g. Gibbs random fields:

- Ergodic properties: dH, Keane, Serafin & Steif 2003.
- Scenery recovery: Löwe & Matzinger 2003.
- Bad configurations: dH, Steif & van der Wal 2004.

This area is largely open and very challenging.