PHASE TRANSITIONS FOR INTERACTING DIFFUSIONS

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§ INTRODUCTION

Consider the following system of interacting diffusions:

$$dX_i(t) = \sum_{j\in\mathbb{Z}^d} a(i,j)[X_j(t) - X_i(t)]\,dt + \sqrt{bX_i(t)^2}\,dW_i(t), \qquad i \in \mathbb{Z}^d,\ t \geq 0.$$

Here,

(1) $a(\cdot,\cdot)$ is an irreducible random walk kernel on $\mathbb{Z}^d \times \mathbb{Z}^d$.

(2) $b \in (0,\infty)$ is a noise parameter.

(3) $\{W_i\}_{i\in\mathbb{Z}^d}$ is a collection of independent standard Brownian motions.

As initial condition we take

$$\{X_i(0)\}_{i \in \mathbb{Z}^d}$$

to be a shift-invariant and shift-ergodic random field with mean

$$\mathbb{E}(X_0(0)) = \theta \in (0, \infty).$$

The evolution preserves this mean.

The above system was studied in detail in

R.A. Carmona and S.A. Molchanov, *Parabolic Anderson Model and Intermittency*, AMS Memoirs 518, 1994.

In particular, the annealed Lyapunov exponents were analyzed as a function of $a(\cdot,\cdot)$ and b.

In the present talk we focus on the ergodic behavior.

All systems with a subquadratic diffusion function satisfy a simple dichotomy. Let $\widehat{a}(i,j) = \frac{1}{2}[a(i,j) + a(j,i)]$, $i,j \in \mathbb{Z}^d$, denote the symmetrized kernel.

(R) If $\widehat{a}(\cdot,\cdot)$ is recurrent, then the system locally dies out.

(T) If $\widehat{a}(\cdot,\cdot)$ is transient, then the system converges to an equilibrium with all moments finite.

In contrast, the PAM has a much richer behavior, due to the fact that the noise term is of the same order of magnitude as the interaction term, resulting in an interesting competition.

§ MOTIVATION

The Parabolic Anderson Model serves as a testing ground for developing techniques that will hopefully be applicable to a large class of systems for which the ergodic behavior remains unsolved.

Key examples are spatial branching models with various types of competition.

§ MAIN THEOREMS

Theorem 1:

(A) If $\widehat{a}(\cdot,\cdot)$ *is recurrent, then the system locally dies out for any* $b > 0$.

(B) If $\widehat{a}(\cdot,\cdot)$ *is transient, then there exist* $b_* > b_2 > 0$ *such that:*

(B1) If $b > b_*$, *then the system locally dies out.*
(B2) If $0 < b < b_*$, *then the system converges to an equilibrium* ν_θ, *which has mean* θ, *is associated and is mixing.*
(B3) ν_θ *has finite* 2-*nd moment if and only if* $0 < b < b_2$.

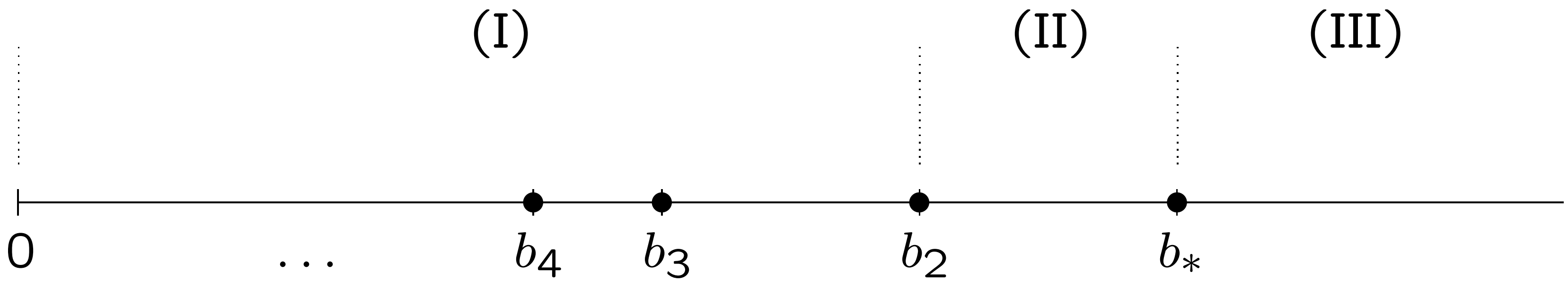
Theorem 2:

(C) If $a(\cdot,\cdot)$ *is transient and symmetric, then there exist* $b_2 \geq b_3 \geq b_4 \geq \cdots > 0$ *such that:*

(C1) ν_θ *has finite* m*-th moment if and only if* $0 < b < b_m$.

(C2) $b_2 \leq (m-1)b_m < 2$ *for all* m.

(C3) $\lim_{m\to\infty}(m-1)b_m = c$ *exists.*



Three regimes:

(I) low noise
(II) moderate noise
(III) high noise.

Conjecture 3:

(D) *The system locally dies out at* $b = b_*$.
(E) $b_2 > b_3 > b_4 > \dots$

We are able to prove strict inequality for special choices of $a(\cdot,\cdot)$, e.g. $b_2 > b_3 > \cdots > b_m$ when the average number of returns to the origin is $\leq 1/(m-2)$.

The latter is true for $m = 3$ and simple random walk in $d \geq 3$.

§ REPRESENTATION FORMULA

The starting point for the analysis is the following representation formula, which is due to Shiga (1992) and is valid when $X(0) \equiv \theta$:

$$X_i(t) = \theta\, e^{-\frac{1}{2}bt}\, \mathbb{E}_i^{\xi}\Big(\exp\Big[\sqrt{b} \int_0^t \sum_{j \in \mathbb{Z}^d} 1_{\{\xi(t-s)=j\}} dW_j(s)\Big]\Big),$$

Here, $\xi = (\xi(t))_{t \geq 0}$ is the random walk with kernel $a(\cdot,\cdot)$ and jump rate 1, and the expectation is over ξ conditioned on $\xi(0) = i$ (ξ and W are independent).

The representation formula leads to the relation, valid for $m \geq 2$,

$$\mathbb{E}^W\Big([X_0(t)]^m \mid X(0) \equiv \theta\Big) = \theta^m \, \mathbb{E}^{\xi^{(m)}}\Big(\exp\Big[bT^{(m)}(t)\Big]\Big),$$

where $\xi^{(m)} = (\xi_1, \ldots, \xi_m)$ are m independent copies of ξ, all starting from 0, and

$$T^{(m)}(t) = \sum_{1 \leq k < l \leq m} T_{kl}(t),$$
$$T_{kl}(t) = \int_0^t 1_{\{\xi_k(s) = \xi_l(s)\}} \, ds,$$

is their intersection local time (in pairs) up to time t.

The above observations, together with several comparison, duality and large deviation techniques, lead to:

(1) For $m \geq 2$,

$$b_m = \sup\Big\{b\colon\ \mathbb{E}^{\xi^{(m)}}\Big(\exp[bT^{(m)}(\infty)]\Big) < \infty\Big\}.$$

(2) $b_* \geq b_{**}$ with

$$b_{**} = \sup\Big\{b\colon\ \mathbb{E}^{\xi_1}\Big(\exp[bT^{(2)}(\infty)] \mid \xi_2\Big) < \infty\ \xi_2 - a.s.\Big\}.$$

Ad (1):

With the help of spectral theory, a variational expression can be derived for b_m, $m \geq 2$, which yields sharp bounds.

The theory of quasi-stationary distributions can be invoked to show that explosion occurs also at b_m.

Ad (2):

With the help of Palm theory, a representation can be derived for b_* in terms of a system that is linearly biased through an auxiliary random walk.

The proof that $b_{**} > b_2$ is rather delicate, since it relies on a new large deviation principle for the empirical process of words read off from a random sequence of letters by a renewal process with polynomial tails.