A variational model for urban planning with traffic congestion

Guillaume CARLIER [a], Filippo SANTAMBROGIO [b]

July 11, 2006

Vancouver, Summer School, July 2006.

[a]University Paris Dauphine
[b]SNS, Pisa

Introduction

Ω : city, open bounded connected subset of $\mathbb{R}^2$ with a smooth boundary, μ, ν: probability measures on $\overline{\Omega}$, with:

μ : distribution of residents (or consumption), ν : distribution of services (or production).

Three effects to be taken into account :

- transportation costs,
- residents are better off with a dispersed μ,
- producers are better off with a concentrated ν (externalities say)

Variational (toy) planning model:

$$\inf_{(\mu,\nu)\in\mathcal{M}_1^+(\overline{\Omega})^2} C(\mu,\nu) + G(\mu) + H(\nu)$$

with C a transportation cost term (taking into account congestion effects), G a functional penalizing concentration and H a functional penalizing dispersion.

Example: C =Wasserstein-like distance, and H with discrete measures as domain (entropy say), G with absolutely continuous measures as domain (Buttazzo-Santambrogio).

In the sequel, very simple choices of G and H:

$$G(\mu) = \begin{cases} \int_\Omega u^2 & \text{if } \mu = u \cdot \mathcal{L}^2,\ u \in L^2(\Omega), \\ +\infty & \text{otherwise;} \end{cases}$$

and

$$H(\nu) := \int_{\overline{\Omega} \times \overline{\Omega}} V(x,y)(\nu \otimes \nu)(dx, dy).$$

H is then an interaction-like term (e.g $V(x,y)$ increasing function of $|x-y|$.)

Plan of the talk

① Congestion

② Optimality conditions

③ Regularity and qualitative properties

④ Examples

Congestion

Beckmann (1952): "A continuous model of transportation": a traffic flow field, i.e. a vector field $\mathbf{Y} : \Omega \to \mathbb{R}^2$ whose direction indicates the consumers' travel direction and whose modulus $|\mathbf{Y}|$ is the intensity of traffic (stationnary, Eulerian). Local equilibrium: in a subregion $K \subset \Omega$ the outflow of consumers equals the excess demand of K:

$$\int_{\partial K} \mathbf{Y} \cdot n \, dS = (\mu - \nu)(K).$$

this formally yields:

$$\mathrm{div}\mathbf{Y} = \mu - \nu. \tag{1}$$

together with the boundary condition (isolated city):

$$\mathbf{Y} \cdot n = 0 \text{ on } \partial\Omega. \tag{2}$$

If transportation cost per consumer is assumed to be uniform, then one may define the transportation cost between μ and ν as the value of the *minimal flow* problem:

$$\inf \left\{ \int_\Omega |\mathbf{Y}(x)| dx \,:\, \mathbf{Y} \text{ satisfies (1)-(2)} \right\}.$$

In fact (convex duality) the previous infimum equals the 1-Wasserstein distance between μ and ν:

$$W_1(\mu, \nu) = \inf\{ \int_{\overline{\Omega}^2} |x - y| d\gamma(x, y) \,:\, \gamma \text{ transport plan}\}$$

(γ transport plan meaning that γ has μ and ν as marginals).

congestion effects: more realistic to assume that the transportation cost per consumer at a point x depends on the intensity of traffic at x itself, $g : \mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing, and assume that if the traffic flow is $\mathbf{Y}$ then the transportation cost per consumer at x is $g(|\mathbf{Y}(x)|)$. It defines the transportation cost between μ and ν as:

$$C_g(\mu, \nu) := \inf \left\{ \int_\Omega g(|\mathbf{Y}(x)|)|\mathbf{Y}(x)|dx \, : \, \mathbf{Y} \text{ satisfies (1)-(2)} \right\}.$$

For the sake of simplicity, we will assume, from now on, that $g(t) = t$ for all $t \in \mathbb{R}_+$, and define the cost:

$$C(\mu, \nu) := \inf \left\{ \int_\Omega |\mathbf{Y}(x)|^2 dx \, : \, \mathbf{Y} \text{ satisfies (1)-(2)} \right\}. \tag{3}$$

$$X := \left\{ \phi \in H^1(\Omega) : \int_\Omega \phi = 0 \right\}.$$

X is a Hilbert space, when equipped with the following inner product and norm:

$$\langle \phi, \psi \rangle_X := \int_\Omega \nabla\phi \cdot \nabla\psi, \ \|\phi\|_X^2 := \langle \phi, \phi \rangle_X .$$

As usual, identify X and its dual X', for every $f \in X'$, there exists, unique, $\phi \in X$ such that:

$$\langle \phi, \psi \rangle_X = f(\psi) \text{ for all } \psi \in X. \tag{4}$$

Note that this implies: $\|f\|_{X'} = \|\phi\|_X$ and we shall also write (4) in the form:

$$\begin{cases} -\Delta\phi = f & \text{in } \Omega, \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega, \phi \in X. \end{cases} \tag{5}$$

With those definitions in mind, our cost functional given by (3) may also be written as:

$$C(\mu, \nu) = \begin{cases} \|\mu - \nu\|_{X'}^2 & \text{if } \mu - \nu \in X', \\ +\infty & \text{otherwise.} \end{cases} \tag{6}$$

To sum up, the planner's (toy) program is:

$$\inf_{(\mu,\nu)\in\mathcal{M}_1^+(\overline{\Omega})^2} F(\mu,\nu) := C(\mu,\nu) + G(\mu) + H(\nu)$$

with:

$$G(\mu) = \begin{cases} \int_\Omega u^2 & \text{if } \mu = u \cdot \mathcal{L}^2,\ u \in L^2(\Omega), \\ +\infty & \text{otherwise;} \end{cases}$$

$$H(\nu) := \int_{\overline{\Omega}\times\overline{\Omega}} V(x,y)(\nu\otimes\nu)(dx,dy).$$

$$C(\mu,\nu) = \begin{cases} \|\mu-\nu\|^2_{X'} & \text{if } \mu-\nu \in X', \\ +\infty & \text{otherwise.} \end{cases}$$

Existence is not a problem (provided V is l.s.c., bdd from below and F is not identically $+\infty$).

The problem is not convex but it is in μ for fixed ν.

Question : regularity of minimizers (all we know a priori is that $\mu \in L^2$ and $\nu \in \mathcal{M}_1^+(\overline{\Omega}) \cap X'$).

Optimality conditions

For fixed $\nu \in \mathcal{M}_1^+(\overline{\Omega}) \cap X'$, minimizing $C(\mu,\nu) + G(\mu)$ over $\mathcal{M}_1^+(\overline{\Omega}) \cap L^2$ yields the unique solution: $\mu = \phi \cdot \mathcal{L}^2$, where $\phi \in H^1(\Omega)$ is the solution of:

$$\begin{cases} -\Delta\phi + \phi = \nu & \text{in } \Omega, \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

we can then reformulate the problem in terms of ν only:

$$J(\nu) := \inf \left\{ F(\mu,\nu) \, : \, \mu \text{ probability measure on } \overline{\Omega} \right\}.$$

we then have:

$$J(\nu) = \begin{cases} \|\phi\|^2_{H^1(\Omega)} + H(\nu) & (\phi \text{ the solution of (7)) if } \nu \in X', \\ +\infty & \text{otherwise.} \end{cases}$$

Identifying $H^1(\Omega)$ and its dual $H^1(\Omega)'$ for its usual Hilbertian structure:

$$\langle \phi, \psi \rangle_{H^1(\Omega)} := \int_\Omega (\nabla\phi \cdot \nabla\psi + \phi\psi),$$

we may also rewrite J as:

$$J(\nu) = \begin{cases} \|\nu\|^2_{H^1(\Omega)'} + H(\nu) & \text{if } \nu \in H^1(\Omega)', \\ +\infty & \text{otherwise.} \end{cases}$$

Finally, the minimization problem in ν reads as:

$$\inf \left\{ J(\nu) \, : \, \nu \text{ probability measure on } \overline{\Omega} \right\}. \tag{8}$$

In what follows, for every $\nu \in H^1(\Omega)'$, we will say that $\phi \in H^1(\Omega)$ is the *potential* of ν if:

$$\langle \phi, \psi \rangle_{H^1(\Omega)} = \nu(\psi), \text{ for all } \psi \in H^1(\Omega). \tag{9}$$

Put differently, the potential of ν is the weak solution of:

$$\begin{cases} -\Delta\phi + \phi = \nu & \text{in } \Omega, \\ \frac{\partial \phi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us also remark that if, in addition, ν is a probability measure on $\overline{\Omega}$ and ϕ its potential, then $\phi \cdot \mathcal{L}^2$ is a probability measure on $\overline{\Omega}$ as well.

Setting:

$$\mathcal{C} = \mathcal{M}_1^+(\overline{\Omega}) \cap H^1(\Omega)' = \left\{ \nu \in H^1(\Omega)' : \nu \geq 0 \text{ in } H^1(\Omega)', \, \nu(1) = 1 \right\}. \tag{10}$$

our aim is to study the problem:

$$\inf_{\nu \in \mathcal{C}} J(\nu) := \|\nu\|^2_{H^1(\Omega)'} + \int_{\overline{\Omega} \times \overline{\Omega}} V(x,y)(\nu \otimes \nu)(dx, dy).$$

In general the quadratic functional J is not convex over $\mathcal{C}$, however it is in the *small* case, i.e. when either V or Ω is small.

Optimality conditions

Assume that $V \in C^0(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})$, set $V^s(x,y) := (V(x,y) + V(y,x))/2$. Given $\nu \in \mathcal{C}$, let ϕ be the potential of ν and let T^s_ν be defined, for all $x \in \Omega$, by:

$$T^s_\nu(x) := \nu(V^s(x,.)) = \int_{\overline{\Omega}} V^s(x,y)\nu(dy).$$

If ν is a solution of (8), then there exists a constant m such that:

$$\phi + T^s_\nu \geq m, \; \phi + T^s_\nu = m \;\; \nu\text{-a.e.}. \tag{11}$$

Regularity

Under the assumption:

Vdiod *(V depends increasingly on distances)*: V is a function of the form $V(x, y) = v(|x - y|^2)$ for a C^2 strictly increasing function v with $v'(s) > 0$ for $s > 0$.

one has an L^∞ estimates in the convex case

Theorem 1 *Suppose that Ω is a bounded, regular and strictly convex open subset of $\mathbb{R}^2$ and that* **Vdiod** *holds. Then, every minimizer $\bar{\nu}$ of J is an absolutely continuous measure with an L^∞ density.*

Idea of the proof= approximation: fix a minimizer $\bar{\nu}$ of J,

$$J_\varepsilon(\nu) = J(\nu) + \varepsilon W_2^2(\nu, \nu_\varepsilon) + \delta_\varepsilon \|\nu\|^2_{L^2(\Omega)},$$

$(\nu_\varepsilon)_\varepsilon$, a sequence of absolutely continuous measures with a strictly positive density, approximating $\bar{\nu}$ in the W_2 distance, and δ_ε is a small parameter ensuring minimizers of J_ε converge to $\bar{\nu}$.

Write down the optimality condition for the approximated problem:

$$\bar{\nu}_\varepsilon = \frac{1}{\delta_\varepsilon}\left(c_\varepsilon - \phi_\varepsilon - T_{\bar{\nu}_\varepsilon} - \frac{\varepsilon}{2}\psi_\varepsilon\right)_+ .$$

get an uniform estimate by maximum principle type arguments and let $\varepsilon \to 0^+$.

This implies that the corresponding μ is $W^{2,p}$ for all p hence $C^{1,\alpha}$ too. In the latter case, we also have:

Proposition 1 *The L^∞ density of any optimal measure ν coincides almost everywhere in* $\operatorname{spt}\nu$ *with a continuous function.*

Under special assumptions, we also have some qualitative properties:

Proposition 2 *Suppose, that $V = V(x-y)$ with V strictly convex. Then the support of ν has non-empty interior.*

Proposition 3 *Suppose, that $V = V(x-y)$ with V strictly subharmonic, i.e. $\Delta V > 0$. Then the support of ν is simply connected.*

Weaker regularity holds in the case of a non convex domain Ω. Let us write $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \partial\Omega \cap \partial\,(\mathrm{co}\,\Omega)$ and $\Gamma_2 = \partial\Omega \setminus \partial\,(\mathrm{co}\,\Omega)$.

Theorem 2 *Suppose that* Γ_1 *is a strictly convex regular boundary and that* **Vdiod** *holds. Then any optimal measure* $\bar{\nu}$ *for* J *can be expressed as* $\bar{\nu} = \bar{\nu}^a + \bar{\nu}^s$, *with* $\bar{\nu}^a \in L^\infty(\Omega)$ *and* $\bar{\nu}^s$ *a singular measure supported on* $\overline{\Gamma}_2$.

Examples

- The unidimensional case: uniqueness provided $V = V(x - y)$ is convex (displacement convexity arguments and convexity properties of the Green function),
- The case of a (small) ball and $V(x, y) = |x - y|^2$, explicit radial solution,
- The case of a (small) crown $B_2 \setminus B_1$ and $V(x, y) = |x - y|^2$: the optimal ν has a singular part on ∂B_1.