Decision problems, curvature and topology

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Decision Problems

Γ a finitely presented group, M a compact manifold, K finite simplicial complex.

1. $\exists$? algorithm that can determine whether or not $\Gamma \cong 1$?
2. Can you decide if $M \cong \mathbb{S}^d$?
3. Is K a manifold?

1. Does M have a finite-sheeted cover?
2. $\exists$? non-trivial $\rho : \Gamma \to \mathrm{GL}(d, K)$?
3. Given $\Gamma < \mathrm{SL}(d, \mathbb{Z})$ can I calculate $H_1(\Gamma, \mathbb{Z})$?

"Given": Answer to last question is YES if finite presentation given, NO if only generators are given (B-Wilton)

Developability and Peter Cameron's Conjecture (2004)

Question

Can one decide if a finite set of partial permutations of a finite set can be extended to permutations of a larger finite set, respecting composition ?

Given partial permutations $p_1, ..., p_m$ of a finite set X (that is, bijections between subsets of X) such that

1. $p_1 = \mathrm{id}_X$, and
2. for all i, j with $\mathrm{dom}(p_i) \cap \mathrm{ran}(P_j) \neq \emptyset$, there is at most one k such that p_k extends $p_i \cdot p_j$

decide whether or not $\exists$ finite set $Y \supseteq X$ and permutations f_i of Y extending the p_i so that if p_k extends $p_i \cdot p_j$ then $f_i \circ f_j = f_k$.

This developability problem can be recast in the language of (rigid) pseudo-groups, groupoids, inverse semigroups, etc., etc.

Aside: What does "Undecidable" mean?

$S \subset \mathbb{N}$ is recursively enumerable (r.e.) if $\exists$ Turing machine that lists S.
And S is recursive if both S and $\mathbb{N} \smallsetminus S$ are r.e.

Proposition

There exist r.e. sets of integers S that are not recursive.

Proposition (=)

There exist $S \subset \mathbb{N}$ for which membership is undecidable.

1. Ability to list S and check that any individual number is in list
2. YES answer can be obtained without problem
3. definitive NO answer is unobtainable

Translation to Groups

Proposition

If $S \subset \mathbb{N}$ is r.e. not recursive, then the word problem is unsolvable in $G = \langle a, b, t \mid \; t\,(b^n a b^{-n}) = (b^n a b^{-n})\, t \;\forall n \in S \rangle$.
(Set of words in the generators that equal $1 \in G$ is r.e. but not recursive).

Can't answer *"does this word $w = w(a, b)$ commute with t?"*

Theorem (Higman Embedding 1961)

Every recursively presented G is a subgroup of a finitely presented group.

Corollary

$\exists$ *finitely presented groups with unsolvable word problem.*

Theorem (Triviality Problem)

$\nexists$ *algorithm to determine whether or not $\Gamma \cong 1$*

Translation to Manifolds: what language to use?

1. integers
2. finite strings over finite alphabets (e.g. group presentations)
3. integer matrices
4. finite simplicial complexes

Naive searches and partial algorithms

Recall YES answer for membership of a r.e. $S \subset \mathbb{N}$ was fine, NO answer was impossible

1. Word problem for finitely presented $\Gamma = \langle A \mid R \rangle$: can naively find YES answer for membership of

$$\{w \in F(A) \mid w = 1 \text{ in } \Gamma\}$$

2. A naive search will always find an isomorphism between a pair of finitely presented groups $\langle A_1 \mid R_1 \rangle$ and $\langle A_2 \mid R_2 \rangle$ if it exists
3. Can find a combinatorial equivalence between finite simplicial complexes K_1, K_2, if it exists
4. (by diagonalising) if K is equivalent to at least one L_i from a list (recursive enumeration)

$$L_1, L_2, \ldots, L_n, \ldots$$

then one can find $K \simeq L_m$ by a naive search

Two stupidities are enough

Two complementary naive searches (complete partial algorithms) will give an algorithm. For example,

Proposition

In any class $\mathfrak{P}$ of finitely presented, residually-finite groups, $\exists$ algorithm to decide $\Gamma \stackrel{?}{=} 1$

Manifold and sphere recognition [Novikov]

Theorem (A)

For each integer $d \geq 5$, there does not exist an algorithm that, given a finite PL triangulation of a closed d-manifold M, can determine whether or not M is homeomorphic to the d-sphere.

Theorem (B)

For each integer $d \geq 6$, there does not exist an algorithm that, given a finite simplicial complex K, can determine whether or not K is homeomorphic to a d-manifold.

Theorem B is easily deduced from Theorem A

$\exists$ algorithm for $d = 3$.

The recognition for $\mathbb{S}^4$ is **open**; it reduces to triviality problem for groups with **balanced presentations** $\langle a_1, \ldots, a_n \mid r_1, \ldots, r_n \rangle$.

Easy direction (naive enumeration)

Let $\mathbb{S}^d = \partial\Delta_{d+1}$

Lemma

There is a partial algorithm that, given a finite simplicial complex L, will correctly identify that L is combinatorially equivalent to $\mathbb{S}^d$ if that is the case, but might not halt otherwise.

Lemma

There is a partial algorithm that, given a finite simplicial complex K of dimension n, will correctly identify if K is a PL triangulation of an n-manifold, but might not halt if K is not such a triangulation.

Wrong approach to sphere recognition

Given a finite presentation P for a group Γ, apply one of the standard ways of fattening P into a closed manifold M_P with $\pi_1 M_P \cong \Gamma$, argue that this can be done algorithmically and claim that $M_P \cong \mathbb{S}^d$ iff $\Gamma \cong 1$.

Better approach (and paradigm)

Vague

1. Think about who the serious characters are and throw the rest away.
2. List all of the plausible candidates
3. Argue that special object can be distinguished from rest of list.
4. Argue that remaining objects harbour as much complexity as arbitrary objects (some translation theorem)

For Theorem A:

1. Restrict attention to homology spheres and perfect groups.
2. Make a list of all homology d-spheres.
3. Poincaré conjecture says $\mathbb{S}^d$ is characterised by $\pi_1 M = 1$.
4. Replace each perfect group by its universal central extension.

Lemma (The list of serious candidates)

For any d, $\exists$ recursive (L_n), finite simplicial d-complexes,

1. *each L_n is PL-triang'n of closed d-manifold, $H_*(L_n, \mathbb{Z}) \cong H_*(\mathbb{S}^d, \mathbb{Z})$;*
2. *every smooth, closed M^d with $H_*(L_n, \mathbb{Z}) \cong H_n(\mathbb{S}^d, \mathbb{Z})$ is homeo to some $|L_n|$ (but no promise of uniqueness).*

Theorem (Kervaire)

If $d \geq 5$, every fin pres Γ with $H_1(\Gamma, \mathbb{Z}) = H_2(\Gamma, \mathbb{Z}) = 0$ is $\pi_1 L_n$, some n.

NB: Only need existence, not construction.

Propn (serious candidates harbour full complexity)

$\exists$ algorithm that replaces a finite presentation of a f.p. perfect group G with a finite presentation of its universal central extension $\tilde{G}$ (without figuring out what G is), and $H_1(\tilde{G}, \mathbb{Z}) = H_2(\tilde{G}, \mathbb{Z}) = 0$.

Thm A: No sphere recognition for $d \geq 5$

List of all homology d-spheres (with repetition)

$$L_1, L_2, \ldots, L_n, \ldots$$

From 2-skeleton we get finite presentations

$$P_1, P_2, \ldots, P_n, \ldots$$

Suppose now that you are given an arbitrary finite presentation $\mathcal{Q}$ of a perfect group G.
Modify it so that you have a presentation of universal central extension $\tilde{G}$, then go along list naively looking for i so that $|P_i| \cong \tilde{G}$.
Kervaire promises that you will find P_i, and the Poincaré conjecture (Smale) says $\tilde{G} \cong 1$ if and only if $L_i \cong \mathbb{S}^d$.

Cannot decide $G \stackrel{?}{=} 1$, so cannot decide $L_i \stackrel{?}{\cong} \mathbb{S}^d$. □

NB: Did not attempt to build a manifold from $\mathcal{Q}$, instead we modified, listed and searched

Switch of focus

Classical

$$\Gamma \overset{?}{\cong} 1$$

Profinite

$$\widehat{G} \overset{?}{\cong} 1$$

Residual Finiteness and Profinite Completion

Γ is residually finite if

$$\forall \gamma \in \Gamma \smallsetminus \{1\} \quad \exists\, \pi : \Gamma \to \text{Finite}, \quad \pi(\gamma) \neq 1$$

Profinite Completion:

$$\hat{\Gamma} := \varprojlim \Gamma/N \qquad |\Gamma/N| < \infty$$

$$\mathcal{F}(\Gamma) := \{\text{isom classes of finite } Q \text{ with } \Gamma \twoheadrightarrow Q\}$$

For Γ_1, Γ_2 finitely generated, $\hat{\Gamma}_1 \cong \hat{\Gamma}_2$ iff $\mathcal{F}(\Gamma_1) = \mathcal{F}(\Gamma_2)$

$\hat{\Gamma} \cong 1$ iff Γ has no finite quotients $(\neq 1)$

For words in the generators of Γ,

$w = 1$ in $\hat{\Gamma}$ iff $w = 1$ in every finite Γ/N

Three Problems. Joint work with Henry Wilton

Question (1. Profinite Triviality)

Does there exist an algorithm that, given a finitely presented group Γ, can determine whether or not Γ has a non-trivial finite quotient?

Question (2. Profinite Isomorphism)

Given a pair of finitely presented, residually finite groups $u : P \hookrightarrow \Gamma$, can one decide if $\hat{u} : \hat{P} \hookrightarrow \hat{\Gamma}$ is an isomorphism? Or if $\hat{P} \cong \hat{\Gamma}$?

Question (3. Cameron's Conjecture)

Can one decide if a finite set of partial permutations of a finite set can be extended to permutations of a larger finite set, respecting composition ?

PLAN:

1. Reduce Questions 2 and 3 to refinements of Question 1.
2. Prove that all of these problems are undecidable.

Universal group of a permutoid

Definition

A permutoid $(\Pi; X)$ is a set Π of partial permutations of a set X such that

1. Π contains 1_X, the identity map of X;
2. for all $p, q \in \Pi$ there exists at most one $r \in \Pi$ such that r extends $p \cdot q$ (if the partial composition exists).

The universal group of a permutoid $(\Pi; X)$ is (cf. Stallings, Baer)

$$\Gamma(\Pi; X) := \langle \Pi \mid pq = r \text{ if } r \text{ extends } p \cdot q \rangle.$$

Lemma

If $(\Pi; X)$ is developable then $\Gamma(\Pi; X)$ has a finite quotient.

Cameron permutoids

$G = \langle A \mid R \rangle$ a finitely presented group, $\rho \in \mathbb{N}$.
$B_\rho \subset G$ ball of radius ρ about $1 \in G$, and $p_1 = \mathrm{id}$ on $B_{2\rho}$.
For $b \in B_\rho \setminus \{1\}$ define $p_b : B_\rho \to B_{2\rho}$ by $p_b(x) = bx$.

Lemma

1. $\mathcal{B}_\rho := (\Pi_\rho; B_{2\rho})$ *is a permutoid, where* $\Pi_\rho = \{p_b \mid b \in B_\rho\}$.
2. *There is a natural quotient map* $\Gamma(\mathcal{B}_r) \to G$, *given by* $p_b \mapsto b$.
3. $\Gamma(\mathcal{B}_r) \cong G$ *if* r *exceeds half the length of the longest relation in* R.
4. *If* $\mathcal{B}_r$ *is developable, then* $\Gamma(\mathcal{B}_r)$ *has a finite quotient.*

Remark: Given $G = \langle A \mid R \rangle$ and $\rho > 0$, one needs to be able to solve the word problem in G in order to construct $\mathcal{B}_\rho$.

Cameron's Conjecture & strong form of Profinite triviality

Proposition

Let $\mathfrak{P}$ be a class of finite presentations for groups in a class where there is a uniform solution to the word problem. If there were an algorithm that could determine which finite permutoids were developable, then there would be an algorithm that could decide for which $\mathcal{P} \in \mathfrak{P}$, the group $P = |\mathcal{P}|$ had a non-trivial finite quotient, i.e. $\widehat{P} = 1$

cf. strategy for sphere recognition

Remark: In proof, one considers non-Cameron permutoids.

Profinite Iso Problem $\widehat{\Gamma_1} \overset{?}{\cong} \widehat{\Gamma_2}$ for residually finite groups

The Bridson-Grunewald construction of Grothendieck Pairs combined with the Algorithmic 1-2-3 Theorem of [B-Howie-Miller-Short] gives algorithm

INPUT: A finite $K(Q,1)$ with $H_1(Q,\mathbb{Z}) = H_2(Q,\mathbb{Z}) = 0$

OUTPUT: A pair of finitely presented groups $u : P \hookrightarrow \Gamma$ with $\Gamma < \mathrm{SL}(d,\mathbb{Z})$, such that $\widehat{P} \cong \widehat{\Gamma}$ (and $\hat{u}$ is iso) iff $\widehat{Q} = 1$.

Enhancing the negative solution to $\hat{\Gamma} \stackrel{??}{\cong} 1$

To resolve Cameron's conjecture on permutoids we need:

Theorem (B-Wilton)

There is a recursive sequence of finitely presented groups G_n, with a uniform solution to the word problem s.t. one can't decide which $\hat{G}_n = 1$.

To resolve the Profinite Isomorphism problem we need

Theorem (B-Wilton)

One can further arrange that $H_1(G_n, \mathbb{Z}) = H_2(G_n, \mathbb{Z}) = 0$, with finite classifying spaces $K(G_n, 1)$ that can be constructed algorithmically.

Arranging a uniform solution to the word problem

Theorem (B-Wilton)

There is no algorithm that, given a compact NPC squared 2-complex X can determine whether or not $\pi_1 X$ has a non-trivial finite quotient (i.e. whether X has a non-trivial finite-sheeted covering).

Fundamental groups of such complexes are biautomatic, and hence there is a uniform solution to the word problem in this class. This finishes the proof of Cameron's conjecture.

IDEA: (cf. Kan-Thurston) Given $\mathcal{P} \equiv \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$, replace the discs in standard 2-complex of $\mathcal{P}$ by copies of NPC squared complexes that have infinite simple π_1.

As $\pi_1 B$ is simple, $\pi_1 X(\mathcal{P})$ and $|\mathcal{P}|$ have the same finite images.

Further refinements for profinite isomorphism problem

Theorem (B-Wilton)

There is a recursive sequence of finite combinatorial CW-complexes K_n,

1. *each K_n is aspherical;*
2. $H_1(K_n, \mathbb{Z}) \cong H_2(K_n, \mathbb{Z}) \cong 0$ *for all $n \in \mathbb{N}$; and*
3. *there is no algorithm to decide for which n we have $\widehat{\pi_1 K_n} \not\cong 1$*

CAT(0) variation on the unsolvability of the Profinite Triviality Problem gives a sequence of 2-complexes like this except $H_2(K, \mathbb{Z})$ infinite.

Remedy this by passing to the universal central extensions. But one needs to control central extension over the blocks B that were added above, so replace B by the standard 2-complex of a group J with $\hat{J} = 1$ that has a balanced aspherical presentation.

One does this in an algorithmic manner and then models the central extensions geometrically with the construction of aspherical torus bundles over the original complexes — more geometry/topology

Why is existence of finite quotients is undecidable?

Consider this sentence Ψ in the first-order theory of groups

$$\forall a, b, c, d : (ba^2b^{-1} \neq a^3) \ \vee \ (dc^2d^{-1} \neq c^3) \ \vee \ ([a,b] \neq d) \\ \vee \ ([c,d] \neq b) \ \vee \ (a = b = c = d = 1).$$

Ψ is true in a group G if and only if there is NO non-trivial homomorphism $B \to G$ where

$$B = \langle a, b, c, d \mid ba^2b^{-1}a^{-3},\ dc^2d^{-1}c^{-3},\ [a,b]d^{-1},\ [c,d]b^{-1} \rangle.$$

Ψ is true in **all groups** iff $B \cong 1$.

Ψ is true in all finite groups iff B has no finite quotients $\neq 1$, i.e. $\hat{B} = 1$

In fact, $B \neq 1$ but $\hat{B} = 1$

Slobodskoi's Theorem

Lemma

If the profinite triviality problem is unsolvable, then the universal theory of finite groups is undecidable.

Theorem (Slobodskoi, 1981)

The universal theory of finite groups is undecidable.

Rough idea of [BW] construction:

- Encode Slobodskoi's construction into a single group G_0
- build a class of groups Γ_w (via controlled Bass-Serre theory) parameterised by words w in the generators of this group
- by constraining possible finite covers of (orbi)spaces associated to these groups, prove that $\hat{\Gamma}_w \cong 1$ if and only if w dies in every finite quotient of the parameter group G_0 (can't decide!)

The heart of BWilton1

One has to work hard to build graphs of groups (and spaces) where the existence of finite quotients can be guaranteed. This involves proving that various subgroups are separable, or malnormal, and exploiting omnipotence (the ability to control relative orders of elements in finite quotients of virtually free groups) etc.

These arguments use geometric and graphical techniques that originate in the work of Stallings and which have been central to Wise's work on special cube complexes which underpins the recent spectacular advances in the understanding of 3-dimensional manifolds.