VERLINDE ALGEBRA

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1. A 2-Dimensional Reduction of Chern-Simons

Quantum Chern-Simons theory is a (3,2,1)-dimensional TFT on oriented manifolds with a p_1-structure with values in the 2-category of $\mathbb{C}$-linear categories,

$$Z_{\mathcal{C}} : \mathrm{Bord}^{(w_1,p_1)}_{\langle 3,2,1\rangle} \to \mathrm{Cat}_{\mathbb{C}},$$

with

$$Z_{\mathcal{C}}(S^1) \simeq \mathcal{C},$$

a modular tensor category. That is, a ribbon fusion category with a non-degenerate S-matrix. In particular, $\mathcal{C}$ is linear (over $\mathbb{C}$), braided, has duals, and is semisimple with finitely many simple objects.

Remark 1.1. A p_1-structure on a manifold M, is the data of a null homotopy of the composition

$$M \to BO \to K(\mathbb{Z}, 4),$$

where $M \to BO$ classifies the (stable) tangent bundle of M, and $BO \to K(\mathbb{Z}, 4)$ is the first Pontryagin class.

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Remark 1.2. Philosophically, a modular tensor category is a categorification of a commutative Frobenius algebra. If $\mathcal{C}$ is a modular tensor category, then $K^0(\mathcal{C})$ inherits the structure of a commutative ring over $\mathbb{Z}$ from the braiding on $\mathcal{C}$, and thus, $K^0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ is a commutative algebra. The trace map $K^0(\mathcal{C}) \to \mathbb{C}$ sends the equivalence class V to $\dim(V) := \mathrm{Trace}(\mathrm{Id}_V)$, the latter of which is defined in any ribbon tensor category.

Definition 1.3. The Verlinde ring is $K^0(\mathcal{C})$, and the Verlinde Algebra of $\mathcal{C}$ is the algebra $K^0(C) \otimes_{\mathbb{Z}} \mathbb{C}$. The former is a Frobenius ring while the latter is a Frobenius algebra.

Two main examples of interest are the following:

Example 1.4. Let G be a finite group, then $\mathrm{Vect}_G(G)$, the category of G-equivariant vector bundles on G, is a modular tensor category. The monoidal structure is defined as follows. Let V be an equivariant vector bundle on G, that is a collection of vector spaces $V_x,\ x \in G$, and isomorphisms $V_x \simeq V_{gxg^{-1}}$ satisfying a cocycle condition. Given two equivariant vector bundles V and W, we define a new equivariant vector bundle $V \otimes^c W$ using convolution:

$$(V \otimes^c W)_x := \bigoplus_{x_1 x_2 = x} V_{x_1} \otimes W_{x_2}.$$

Notice that,

$$\begin{aligned}
(V \otimes^c W)_{gxg^{-1}} &:= \bigoplus_{x_1 x_2 = gxg^{-1}} V_{x_1} \otimes W_{x_2} \\
&\simeq \bigoplus_{g^{-1}x_1 x_2 g = x} V_{x_1} \otimes W_{x_2} \\
&\simeq \bigoplus_{g^{-1}x_1 g g^{-1} x_2 g = x} V_{x_1} \otimes W_{x_2} \\
&\simeq \bigoplus_{g^{-1}x_1 g g^{-1} x_2 g = x} V_{g^{-1}x_1 g} \otimes W_{g^{-1}x_2 g} \\
&\simeq \bigoplus_{y_1 y_2 = x} V_{y_1} \otimes W_{y_2} \\
&\simeq (V \otimes^c W)_x
\end{aligned}$$

So that $V \otimes^c W$ is indeed another equivariant vector bundle. One can show, $K^0(\mathcal{C}) = K_G(G)$ is a Frobenius algebra, multiplication arises from the pushforward of group multiplication.

Example 1.5. There is a twisted version of the example above. Let $\alpha \in H^4(BG, \mathbb{Z}) \to H^3_G(G, \mathbb{Z}) \simeq H^2_G(G, U(1)) \simeq H^1_G(G, \{\text{Line Bundles}\})$ where the first arrow in the sequence sends a map $BG \to B^4\mathbb{Z}$ to a map $G/G \to B^2\mathbb{C}^\times \times B^3\mathbb{C}^\times \to B^2\mathbb{C}^\times$, by taking free loops. That is, we get a $\mathbb{C}^\times$-gerbe on G/G. For the remaining arrows, recall that

$$\mathbb{Z} \simeq K(\mathbb{Z}, 0),\ U(1) \simeq K(\mathbb{Z}, 1),\ \mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2).$$

From the data of α, one can construct hermitian lines $L_{x,y}$ with isomorphisms $L_{yxy^{-1},z} \otimes L_{x,y} \to L_{x,zy}$, where $x, y, z \in G$. Then $\text{Vect}^\alpha_G(G)$, the category of α-twisted equivariant vector bundles on G, is a modular tensor category with a monoidal structure given by α-twisted convolution. An object in this category is a vector bundle V over G together with isomorphisms $L_{x,y} \otimes V_x \to V_{yxy^{-1}}$, where the $L_{x,y}$ are hermitian lines constructed using the data of α. These are equivariant vector bundles twisted by a gerbe.

Definition 1.6. Now, let G be simply connected, compact, simple Lie group and let

$$1 \to \mathbb{C}^\times \to \tilde{LG} \to LG \to 1$$

be the universal central extension corresponding to a generator of $H^2(LG, \mathbb{C}^\times)$.

Remark 1.7. A projective representation of LG is equivalent to a honest representation of $\tilde{LG}$, where we require the center $\mathbb{C}^\times$ to act by scalar multiplication.

Definition 1.8. A positive energy representation of LG at level α is a representation of $\tilde{LG}$, V, extending to the semi-direct product $\tilde{LG} \rtimes Rot(S^1)$, such that $Rot(S^1)$ acts by non-negative characters only. That is, $Rot(S^1)$ induces a decomposition of vector spaces

$$V = \bigoplus_{n \geq 0} V(n)$$

where $V(n) = \{v \in V | R_\theta v = e^{in\theta} v\}$ and $R_\theta \in Rot(S^1)$.

Remark 1.9. If V is irreducible, the kernel of the central extension acts by a single scalar α (Schur's Lemma), called the level of the representation. The level classifies the central extension of LG and is a class $\alpha \in H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$. Furthermore, V is determined by its level and its lowest nonzero energy eigenspace, which itself is an irreducible representation of G. We will use this fact in the sequel.

Proposition 1.10. *Given G and an element $\alpha \in H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$, the category of positive energy representations of the loop group LG at level α, $\mathrm{Rep}^\alpha(LG)$, is a modular tensor category.*

Definition 1.11. Let $Ver_\alpha(G)$ be the Verlinde ring of the modular tensor category $\mathrm{Rep}^\alpha(LG)$, and let $Ver_\alpha(G) \otimes_{\mathbb{Z}} \mathbb{C}$ be its Verlinde algebra.

In this talk, we will consider a 2-dimensional reduction of Chern-Simons theory. This is an oriented 2-dimensional TFT $Z'_{\mathcal{C}}$ defined by

$$Z'_{\mathcal{C}}(M) := Z_{\mathcal{C}}(S^1 \times M).$$

In particular,

$$Z'_{\mathcal{C}}(\mathrm{pt}) := Z_{\mathcal{C}}(S^1 \times \mathrm{pt}) \simeq \mathcal{C}.$$
$$Z'_{\mathcal{C}}(S^1) \simeq \mathrm{HH}_0(\mathcal{C}).$$

Remark 1.12. We consider an oriented 2-dimensional TFT because the map defining the 2-dimensional reduction

$$\mathrm{Bord}_2^{(w_1,p_1)} \xrightarrow{S^1\times -} \mathrm{Bord}_{\langle 3,2,1\rangle}^{w_1,p_1} \longrightarrow \mathrm{Cat}_{\mathbb{C}}$$

factors through the oriented bordism category $\mathrm{Bord}_2^{w_1}$.
We claim there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Bord}_{\langle 3,2,1\rangle}^{(w_1,p_1)} & \longrightarrow & \mathrm{Cat}_{\mathbb{C}} \\ \uparrow{\scriptstyle S^1\times -} & & \vdots \\ \mathrm{Bord}_2^{(w_1,p_1)} & \longrightarrow & \mathrm{Bord}_2^{w_1}. \end{array}$$

Goal 1.13. *Show the Verlinde Algebra $K^0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ is the Frobenius algebra defining the (2,1)-dimensional reduction $Z'_{\mathcal{C}}$.*

Thus, we must show the following

Proposition 1.14. *Let $\mathcal{C}$ be a modular tensor category, then*

$$K^0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathrm{HH}_0(\mathcal{C}).$$

Proof. There is an isomorphism $K^0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ with the algebra of $\mathbb{C}$-valued functions on the finite set I of isomorphism classes of simple objects. This uses the non-degeneracy of the S-matrix [Bakalov-Kirillov 3.1.12]. This algebra can be interpreted as $\mathrm{End}(\mathrm{Id}_{\mathcal{C}})$ by Schur's Lemma. Furthermore, $\mathrm{End}(\mathrm{Id}_{\mathcal{C}}) \simeq \mathrm{HH}_0(\mathcal{C})$, using the semisimplicity of the category $\mathcal{C}$. □

2. The example $G = SU(2)$ and $\alpha = k$

Start with the complexified representation ring $Rep(SU(2)) = \mathbb{C}[t, t^{-1}]^{\Sigma_2}$. That is, the irreducible representations are V_n with $\dim(V_n) = n + 1$. This representation corresponds to the polynomial $t^n + t^{n-2} + \ldots + t^{-n}$. Multiplication of polynomials gives the formula:

$$V_n \otimes V_m = V_{m+n} \oplus V_{m+n-2} \oplus \ldots \oplus V_{|m-n|}.$$

The Verlinda algebra, $Ver_k(SU(2)) \otimes_{\mathbb{Z}} \mathbb{C}$, is a quotient of $Rep(SU(2))$ by

$$V_{k+1} = 0$$

and the relation

$$V_n \oplus V_{2k+2-n} = 0.$$

Example 2.1. Take $k = 5$, then in $Rep(SU(2))$. Draw a picture with a mirror at 6!

$$V_3 \otimes V_4 = V_7 \oplus V_5 \oplus V_3 \oplus V_1$$

and in the quotient $Ver_5(SU(2))$ this becomes

$$V_3 \otimes V_4 = -V_5 \oplus V_5 \oplus V_3 \oplus V_1 = V_3 \oplus V_1.$$

If $k = 0$ we have

$$Ver_0(SU(2)) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[x]/x.$$

If $k = 1$ we have

$$Ver_1(SU(2)) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[x]/(x^2 - 1).$$

One can show:

$$Ver_{k-1}(SU(2)) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[x]/(\prod_{m=1}^{k}(x - 2\cos(\frac{m}{2k+2}2\pi))).$$

Remark 2.2. Again, positive energy representations of level k are determined by their lowest energy eigenstate which itself is an irreducible representation of the group, in this case $SU(2)$. The first equation $V_{k+1} = 0$ corresponds to the fact that the antidominant weights controlling irreducible representations live in the positive Weyl alcove [Segal-Pressley 9.3.5].

Remark 2.3. The unit of the algebra is called the Vacuum representation of level k. It is the positive energy representation of level k that is induced from a lowest energy eigenspace being a lowest weight representation.

Remark 2.4. The "fusion" algebra structure has origins in conformal field theory. Let V_p, V_q be irreducible positive energy representations of $G = SU(2)$. Then,

$$V_p \cdot V_q = \sum_{V_r} N_{V_p,V_q}^{V_r} V_r,$$

Here, $N_{V_p,V_q}^{V_r}$ is the dimension of the vector space

$$(V_p \otimes V_q \otimes V_r^*)^{\mathrm{Hol}(\mathbb{P}^1 - \{p_1,p_2,p_3\}, G_{\mathbb{C}})}.$$

This is (dual to) the space of conformal blocks. One can show this multiplication is associative and gives rise to the Verlinde algebra. There is a subtle point here. Let $\hat{G}$ be the canonical central extension of $\mathrm{Hol}(\mathbb{C}^\times, G_{\mathbb{C}})^{\times 3}$ that extends each of the individual universal central extensions. Then, one needs to show the image of

$$\mathrm{Hol}(\mathbb{P}^1 - \{p_1, p_2, p_3\}, G_{\mathbb{C}}) \to \hat{G}$$

splits to actually have a well defined action of $\mathrm{Hol}(\mathbb{P}^1 - \{p_1, p_2, p_3\}, G_{\mathbb{C}})$ on $V_p \otimes V_q \otimes V_r^*$. This is established by using the residue formula.

Furthermore, one can show that:

$$N_{V_p,V_q}^{V_r} = \begin{cases} 1 & \text{if } r - |p-q| \text{ is even and } |p-q| \leq r \leq \min(p+q, 2k-p-q) \\ 0 & \text{otherwise} \end{cases}$$

This computation is in Verlinde's original paper.

3. Twistings and Orientations

To give a complex vector bundle on M is to give vector bundles V_i on open sets U_i of a covering and isomorphisms

$$\lambda_{ij} : V_i \to V_j$$

which satisfy a cocycle condition on intersections. In complex K-theory this is expressed by the Mayer-Vietoris sequence. In forming a twisted vector bundle V, one introduces a complex line bundle L_{ij} on $U_i \cap U_j$ together with isomorphisms:

$$\lambda_{ij} : L_{ij} \otimes V_i \to V_j.$$

The L_{ij} must come equipped with isomorphisms

$$L_{jk} \otimes L_{ij} \to L_{ik}$$

on triple intersections and satisfy a cocycle condition on quadruple intersections. Thus, we can form a twisted version of $K(M)$ given an element $\tau \in H^1(M, \{\text{Line Bundles}\}) \simeq H^3(M, \mathbb{Z})$. This group parametrizes complex $\mathbb{C}^\times$-gerbes.

Remark 3.1. There is second way to think about this. Recall that $\mathcal{F}$ (the space of Fredholm operators of a complex Hilbert space $\mathcal{H}$) is a representing space for K-theory. That is, $K(X) = \pi_0\Gamma(X \times \mathcal{F} \to X)$. If $U = U(\mathcal{H})$ is the unitary group and $P \to X$ is a principal PU-bundle, one can form the associated bundle $\xi = P \times_{PU} \mathcal{F} \to X$ with fiber $\mathcal{F}$. Define P-twisted K-theory to be

$$K(X)_P = \pi_0\Gamma(\xi \to X).$$

Thus one twists K-theory by PU-bundles over X, and isomorphism classes of such bundles are given by $[X, BPU]$. Since BPU is a model for $K(\mathbb{Z}, 3)$, we arrive at $[X, BPU] \simeq H^3(X, \mathbb{Z})$.

Remark 3.2. A third way to think about this from an ∞-categorical perspective is in Ando-Blumberg-Gepner's "Twists of K-Theory." They discuss a map $K(\mathbb{Z},3) \xrightarrow{T} BGL_1(K) \simeq |\mathrm{Line}_K|$ and form the composition

$$M \xrightarrow{\tau} K(\mathbb{Z},3) \xrightarrow{T} BGL_1(K) \simeq |\mathrm{Line}_K|.$$

The corresponding Thom spectrum is

$$M^{T\tau} := \mathrm{colim}(\mathrm{Sing}M \xrightarrow{T\tau} \mathrm{Line}_K \longrightarrow \mathrm{Mod}_K).$$

Finally, twisted K-theory is given by

$$K^n_\tau(M) := \pi_0(\mathrm{Mod}_K(M^{T\tau}, \Sigma^n K)).$$

Now, further allowing for the L_{ij} defined before the remarks to have a degree modulo 2 and be $(\pm)$ line bundles, we arrive at elements

$$\tau \in H^0(M,\mathbb{Z}/2) \times H^1(M,\mathbb{Z}/2) \times H^3(M,\mathbb{Z}).$$

Remark 3.3. That is, twistings of K-theory on a space M are classified up to isomorphism by the set $H^0(M,\mathbb{Z}/2) \times H^1(M,\mathbb{Z}/2) \times H^3(M,\mathbb{Z})$.

Example 3.4. A *real* vector bundle $V \to M$ determines a twisting τ_V in complex K-theory, whose equivalence class is:

$$[\tau_V] = (\mathrm{rank}V, w_1(V), W_3(V)) \in H^0(M,\mathbb{Z}/2)\times H^1(M,\mathbb{Z}/2)\times H^3(M,\mathbb{Z}).$$

More precisely, a real vector bundle V has a second Stiefel-Whitney class $w_2(V) \in H^2(M,\mathbb{Z}/2)$, and gives a real $\mathbb{R}^\times$-gerbe. The third integral Stiefel-Whitney class $W_3(V)$ is the image of $w_2(V)$ under the Bockstein $H^2(M,\mathbb{Z}/2) \to H^3(M,\mathbb{Z})$, and corresponds to complexification.

One can further assign a twisting to any virtual real vector bundle by setting $\tau_{-V} := -\tau_V$.

Remark 3.5. Let τ be a twisting on a manifold N, then to a proper map $p : M \to N$ one can define a pushforward map

$$p_* : K^{(\tau_p + p^*\tau)+\bullet}(M) \to K^{\tau+\bullet}(N).$$

where $\tau_p = \tau_M - p^*\tau_N$ is the twisting associated to the relative tangent bundle.

Definition 3.6. A KU-orientation of V is an equivalence $\tau_V \simeq \tau_{\underline{\text{rank}V}}$ or equivalently, a trivialization of the twisting attached to the reduced bundle $(V - \underline{\text{rank}V})$. Equivalently, this is a *spin*^{c} structure on V.

Definition 3.7. An orientation of a manifold or stack is an orientation of its (virtual) tangent bundle. Recall that the tangent bundle to a smooth stack is a graded vector bundle. We form a virtual bundle by taking the alternating sum of its homogeneous components.

Definition 3.8. A KU-orientation of a map $p : M \to N$ is a trivialization of the twisting $\tau_{TM - p^*TN - \underline{\text{rank}p}} = \tau_p - \tau_{\underline{\text{rank}p}}$. Thus, to a KU-oriented, proper map $p : M \to N$ one can define a pushforward map

$$p_* : K^{\bullet + \dim(M) - \dim(N)}(M) \to K^\bullet(N).$$

Example 3.9. If X is a closed oriented 2-manifold, the tangent space of the stack M_X at A (a connection on principal bundle P) is the complex

$$0 \to \Omega^0_X(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1_X(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^2_X(\mathfrak{g}_P),$$

where $\mathfrak{g}_P$ is the adjoint bundle associated to P. One forms the virtual tangent bundle to M_X as the index of an elliptic complex. The reduced tangent bundle to M_X is computed by the de Rham complex coupled to the reduced adjoint bundle $\overline{\mathfrak{g}_P} := \mathfrak{g}_P - \underline{\dim G}$.

Freed-Hopkins-Teleman describe a universal orientation that simultaneously orients M_X for not only closed 2-manifolds X, but 2-manifolds with boundary. In particular, the restriction maps $t : M_X \to M_{\partial X}$ are oriented. That is, there is a trivialization of the twisting $\tau_t - \tau_{\underline{\text{rank}t}}$.

4. Pushforward Using Consistent Orientations

Let G be a compact Lie group. Let Z be a 1 or 2 dimensional oriented manifold. Let M_Z be the stack of flat connections of X. For example, $M_{S^1} \simeq G/G$. To a bordism $X : Y_0 \to Y_1$ we consider the correspondence of flat G-connections:

$$\begin{array}{ccccc} & & M_X & & \\ & \swarrow^{s} & & \searrow^{t} & \\ M_{Y_0} & & & & M_{Y_1} \end{array}$$

We would like to define a push-pull

$$Z_X := t_* \circ s^* : K^\bullet(M_{Y_0}) \to K^\bullet(M_{Y_1}).$$

But the pushforward, t_* requires an orientation on (twisted) K-theory. Freed-Hopkins-Teleman show that orientations can be consistently chosen. Moreover, the functor Z respects gluing, i.e. is functorial and defines a 2d-TFT. For instance, given

$$\begin{array}{ccccccc} & & & M_{X'\circ X} & & & \\ & & \swarrow^{r} & & \searrow^{r'} & & \\ & M_X & & & & M_{X'} & \\ \swarrow^{s} & & \searrow^{t} & & \swarrow^{s'} & & \searrow^{t'} \\ M_{Y_0} & & & M_{Y_1} & & & M_{Y_2} \end{array}$$

We have that

$$(t'r')_* \circ (sr)^* = [t'_* \circ s'^*] \circ [t_* \circ s^*]$$

Remark 4.1. Moreover, they show there is a well-defined map from "consistent orientations" to levels on G.

Remark 4.2. Notice that $M_{S^1} \simeq G/G$ as stacks, and thus $K^\bullet(M_{S^1}) \simeq K^\bullet_G(G)$.

Theorem 4.3. *For any compact Lie group G, once a consistent orientation is chosen (and hence a level), the value of S^1 on the corresponding 2d TFT recovers the Frobenius ring*

$$K_G^{\mathfrak{g}+\check{h}+\alpha}(G) \simeq Ver_\alpha(G).$$

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