Problem Set # 1

WCATSS 2014

The daily problem sets are intended to stimulate discussions and to focus your thoughts about the lectures. Not all problems are directly related to the day's lectures; some anticipate future lectures. We encourage you to form groups to discuss the lectures and work together on problems. Be sure to ask questions, both to other students and to the more senior mathematicians around. There may be confusing terminology or notation, unfamiliar concepts, unclear diction, etc. Ask questions! You may ask Google too, but as there are many mathematicians around it will be more fun to engage in conversations. And feel free to pose your own problems—to yourself, your group, the organizers, . . .

The references in the syllabus are a starting point for further explorations. In turn, they have bibliographies that are gateways into the literature.

1. In this problem you'll compute some classical Chern-Simons invariants of *flat* connections on a principal G-bundle over a closed oriented 3-manifold for various G.
   (a) Let $G = \mathbb{Z}/2\mathbb{Z}$ and λ the nonzero class in $H^4(BG;\mathbb{Z})$. Compute the Chern-Simons invariant of the nontrivial double cover of $S^1 \times S^2$. Of the nontrivial double cover of $\mathbb{RP}^3$.
   (b) Let $G = U(1)$ and λ the generator of $H^4(BG;\mathbb{Z})$ which is the square of a class in $H^2(BG;\mathbb{Z})$. Compute the Chern-Simons invariant of the flat G-bundle on $S^1 \times S^1 \times S^1$ whose holonomies around the three coordinate circles are $\lambda_1, \lambda_2, \lambda_3 \in G$. Of the nontrivial flat G-bundle on $\mathbb{RP}^3$?

2. In this problem you'll construct a simple topological field theory $F\colon \mathrm{Bord}_{\langle 0,1\rangle} \to \mathrm{Vect}_{\mathbb{Q}}$ from the bordism category of 0- and 1-manifolds to the category of rational vector spaces. Fix a finite group G. For any manifold M let $\mathcal{C}(M)$ denote the groupoid of principal G-bundles over M.
   (a) Prove that $\mathcal{C}(S^1)$ is equivalent to the groupoid $G/\!/G$ defined by G acting on itself by conjugation. You should spell out precisely what these groupoids are.
   (b) For a compact 0-manifold Y, let $F(Y)$ be the vector space of functions $\mathcal{C}(Y) \to \mathbb{Q}$. Say what you mean by such functors on a groupoid.
   (c) For a *closed* 1-manifold X define

$$F(X) = \sum_{[P]\in\pi_0\mathcal{C}(X)} \frac{1}{\#\,\mathrm{Aut}(P)},$$

   where the sum is over equivalence classes of principal G-bundles. Extend this to all bordisms $X\colon Y_0 \to Y_1$.

(d) Check that F is a symmetric monoidal functor.

(e) Calculate F on the "left elbow"—the connected bordism from the empty 0-manifold $\emptyset^0$ to the 0-manifold consisting of 2 points—and also on the "right elbow". Use these to compute $F(S^1)$.

3. Let $f\colon M \to \mathbb{R}$ be a smooth map from a smooth manifold. Show that f is a Morse function if and only if the section $df\colon M \to T^*M$ of the cotangent bundle is transverse to the zero section. Note that the image of this section is a Lagrangian in T^*M with its standard symplectic structure. (If $x^1, \dots, x^n$ are local coordinates on M, and we write a local 1-form as $p_i dx^i$, then $x^1, \dots, x^n, p_1, \dots, p_n$ are local coordinates on T^*M and the symplectic form is $dp_i \wedge dx^i$.)

4. Consider a standard embedding of a 2-torus T in $\mathbb{R}^3$ so that it is prepared to roll, in a balanced way, along the horizontal xy-plane. The z-coordinate gives a height function $z\colon T \to \mathbb{R}$. Notice that z is Morse, but it is *not* Morse-Smale (which is to say that the stable and unstable manifolds do not intersect transversely). Tilt the torus so this height function, now called $h\colon T \to \mathbb{R}$, is Morse-Smale. Regard T as a Riemannian manifold, with metric inherited from that of $\mathbb{R}^3$.

   (a) List each critical point of h together with its index.

   (b) For each pair (p, q) of critical points, identify the manifold with corners $\mathcal{M}(p, q)$ which is the compactification of the moduli space of gradient flow lines from p to q.

   (c) Explicate the Morse chain complex for h. Try to *see* that the differential squares to zero.