Expansion subshifts of Möbius number systems

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Itineraries

X: a compact metric space,
$F : X \to X$ a continuous mapping,
$\mathcal{W} = \{W_a : \ a \in A\}$ a finite open cover of X.

$u \in A^*$ is an itinerary of $x \in X$ if $F^n(x) \in W_{u_n}$

$$W_u = W_{u_0} \cap F^{-1} W_{u_1} \cap \cdots \cap F^{-n} W_{u_n}, \ u \in A^{n+1}$$

$\mathcal{W}$ is a generator if

$$\lim_{n \to \infty} \max\{\mathrm{diam}(W_u) : \ u \in A^n\} = 0$$

Symbolic extensions

$$\begin{aligned} \mathcal{S}_{\mathcal{W}} &= \{u \in A^{\mathbb{N}} : \ \forall n, W_{u_{[0,n)}} \neq \emptyset\} \\ \{\Phi(u)\} &= \bigcap_{n>0} \overline{W_{u_{[0,n)}}}, \ u \in \mathcal{S}_{\mathcal{W}} \end{aligned}$$

$\Phi : \mathcal{S}_{\mathcal{W}} \to X$ is continuous and surjective.

$$\begin{array}{ccc} \mathcal{S}_{\mathcal{W}} & \xrightarrow{\sigma} & \mathcal{S}_{\mathcal{W}} \\ \Phi \downarrow & & \downarrow \Phi \\ X & \xrightarrow{F} & X \end{array}$$

When is $\mathcal{S}_{\mathcal{W}}$ an SFT or sofic?

Regular open sets

$U \subseteq X$ is regular open if it is the interior of its closure: $U = \overline{U}^{\circ}$.

Regular open sets form a Boolean algebra:
$U^{\perp} = X \setminus \overline{U}$: complement,
$U \vee V = \overline{U \cup V}^{\circ}$: join,
$U \cap V$: meet.

Regular open almost-covers

$\mathcal{W} = \{W_a : a \in A\}$ is an almost-cover of X if, $W_a = \overline{W_a}^{\circ}$ are regular open, and $\bigcup_{a \in A} \overline{W_a} = X$.

$\mathcal{W}$ is a cover if $\bigcup_a W_a = X$.
$\mathcal{W}$ is a partition if $W_a \cap W_b = \emptyset$ for $a \neq b$:

If $\mathcal{W}$, $\mathcal{V}$ are almost-covers then $\mathcal{W} \vee \mathcal{V} = \{W_a \cap V_b : a \in A, b \in B\}$ is an almost-cover.

Each almost-cover $\mathcal{W}$ has a refinement partition $\mathcal{V} = \bigvee_a \{W_a, X \setminus \overline{W_a}\}$ with

$$V_b \cap W_a \neq \emptyset \implies V_b \subseteq W_a$$

Iterative systems: A^*-actions on a compact X

$\mathcal{F} = \{F_a : X \to X\}_{a\in A}$ homeomorphisms
$F_u = F_{u_n} \circ \cdots \circ F_{u_1} \circ F_{u_0}$, $F_{uv} = F_v \circ F_u$
$\mathcal{W} = \{W_a : \ a \in A\}$ is an almost-cover of X.

$u \in A^{n+1}$ is an itinerary of $x \in X$ if $F_{u_{[0,i)}}(x) \in W_{u_i}$:
$x \in W_{u_0}$, $F_{u_0}(x) \in W_{u_1}$, $F_{u_1}F_{u_0}(x) = F_{u_{[0,2)}}(x) \in W_{u_2}$

$$
\begin{aligned}
W_u &= W_{u_0} \cap F_{u_0}^{-1} W_{u_1} \cap \cdots \cap F_{u_{[0,n)}}^{-1} W_{u_n}, \ u \in A^{n+1} \\
W_{uv} &= W_u \cap F_u^{-1} W_v
\end{aligned}
$$

$\forall n, \{W_u : \ u \in A^n\}$ is an almost-cover of X.

Expansion subshift: Assume that $\mathcal{W}$ is a generator

If $F_a : \overline{W_a} \to X$ are expansions then $\mathcal{W}$ is a generator.

$$\lim_{n\to\infty} \max\{\mathrm{diam}(W_u) : \; u \in A^n\} = 0$$

$$\begin{aligned}
\mathcal{L}_{\mathcal{W}} &= \{u \in A^* : \; W_u \neq \emptyset\} \\
\mathcal{S}_{\mathcal{W}} &= \{u \in A^{\mathbb{N}} : \; \forall n, W_{u_{[0,n)}} \neq \emptyset\} \\
\{\Phi(u)\} &= \bigcap_n \overline{W_{u_{[0,n)}}}
\end{aligned}$$

Theorem. $\Phi : \mathcal{S}_{\mathcal{W}} \to X$ is a continuous surjection.

Regular continued fractions

$$X = \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\},\ A = \{\bar{1}, 0, 1\}$$
$$F_{\bar{1}}(x) = x + 1,\ W_{\bar{1}} = (-\infty, 0)$$
$$F_0(x) = 1/x,\ W_0 = (0, 1)$$
$$F_1(x) = x - 1,\ W_1 = (1, \infty)$$

forbidden words: $\{\bar{1}1, 1\bar{1}, 0\bar{1}, 00\}$

$$F_{\bar{1}} W_{\bar{1}} = W_{\bar{1}} \cup W_0,\ F_0 W_0 = W_1,\ F_1 W_1 = W_0 \cup W_1$$
$$u = 1^{a_0} 0 1^{a_1} 0 1^{a_2} \cdots,\ a_0 \in \mathbb{Z},\ a_n > 0 \text{ for } n > 0$$

$$\Phi(u) = [a_0, a_1, \cdots] = a_0 + 1/(a_1 + 1/(a_2 + \cdots$$

$\mathcal{S}_{\mathcal{W}}$ is an SFT of order 2

Binary signed system

$$X = \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\},\ A = \{\bar{1}, 1, 2\}$$
$$F_{\bar{1}}(x) = 2x + 1,\ W_{\bar{1}} = (-1, 0)$$
$$F_1(x) = 2x - 1,\ W_1 = (0, 1)$$
$$F_2(x) = x/2,\ W_2 = (1, -1)$$

$$F_{\bar{1}} W_{\bar{1}} = F_1 W_1 = W_{\bar{1}} \cup W_1,$$
$$F_2 W_2 = (\tfrac{1}{2}, -\tfrac{1}{2}),\ F_1(\tfrac{1}{2}, 1) = (0, 1)$$

forbidden words: $\{\bar{1}2, 12, 21\bar{1}, 2\bar{1}1\}$

$$u = 2^n v,\ v \in \{\bar{1}, 1\}^*,$$

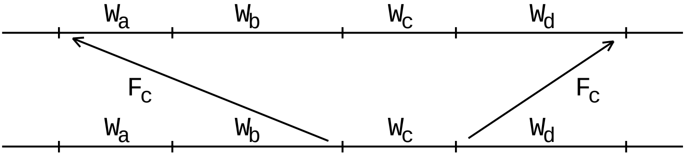
$$\Phi(u) = \sum_{i=0}^{\infty} 2^{n-i} v_i$$

$\mathcal{S}_{\mathcal{W}}$ is an SFT of order 3

Expansion subshifts of finite type

Theorem. $\mathcal{S}_{\mathcal{W}}$ is an SFT of order $m+1$ iff $\forall a \in A, u \in A^m$

$$W_u \cap F_a W_a \neq \emptyset \implies W_u \subseteq F_a W_a$$



Theorem. If X is connected and $\mathcal{W}$ is a cover, then $\mathcal{S}_{\mathcal{W}}$ is not an SFT.

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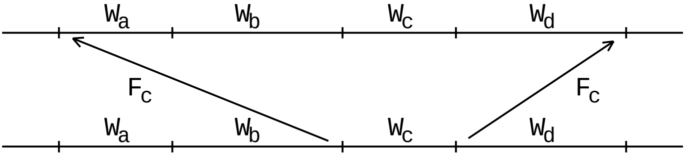


Theorem. If X is connected and $\mathcal{W}$ is a cover, then $\mathcal{S}_{\mathcal{W}}$ is not an SFT.

SFT of order $m+1$: $a \in A$, $u \in A^m$

$$W_u \cap F_a W_a \neq \emptyset \implies W_u \subseteq F_a W_a$$

$$F_a^{-1} W_u \cap W_a \neq \emptyset \implies F_a^{-1} W_u \subseteq W_a$$

$$W_{uv} = W_u \cap F_u^{-1} W_v$$

Let $u \in A^{n+1}$, $\forall i$, $u_{[i,i+m]} \in \mathcal{L}_{\mathcal{W}}$.

$$\begin{aligned} W_u &= W_{u_0} \cap F_{u_0}^{-1} W_{u_{[1,m]}} \cap F_{u_{[0,m]}}^{-1} W_{u_{[m+1,n]}} \\ &= F_{u_0}^{-1} W_{u_{[1,m]}} \cap F_{u_{[0,m]}}^{-1} W_{u_{[m+1,n]}} \\ &= F_{u_0}^{-1} W_{u_{[1,n]}} = \cdots = F_{u_{[0,n-m]}}^{-1} W_{u_{(n-m,n]}} \neq \emptyset \end{aligned}$$

Thus $u \in \mathcal{L}_{\mathcal{W}}$.

Conversely: $a \in A$, $u \in A^m$

Let $W_u \cap F_a W_a \neq \emptyset$ but $W_u \not\subseteq F_a W_a$.

$\exists v, W_v \subseteq F_u W_u \setminus F_{au} W_a = F_u(W_u \setminus F_a W_a)$

$W_{uv} = W_u \cap F_u^{-1} W_v = F_u^{-1} W_v \neq \emptyset$,

$W_{auv} = W_a \cap F_a^{-1} W_u \cap F_{au}^{-1} W_v = \emptyset$

$au \in \mathcal{L}_{\mathcal{W}}$, $uv \in \mathcal{L}_{\mathcal{W}}$, $auv \notin \mathcal{L}_{\mathcal{W}}$

$\mathcal{S}_{\mathcal{W}}$ is not an SFT of order $m+1$

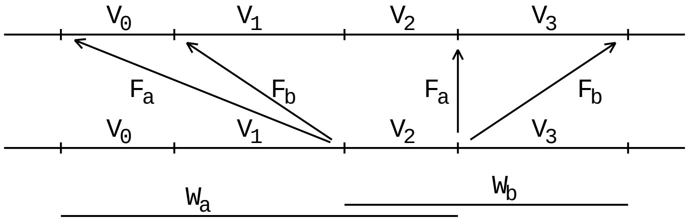
Sofic expansion subshifts

Theorem. $\mathcal{S}_{\mathcal{W}}$ is sofic iff there exists a partition $\mathcal{V} = \{V_p : \ p \in B\}$ with

1. $V_p \cap W_a \neq \emptyset \implies V_p \subseteq W_a$
2. $V_p \subseteq W_a, V_q \cap F_a V_p \neq \emptyset \implies V_q \subseteq F_a V_p$.

Then $\mathcal{S}_{\mathcal{W}}$ is the subshift of the labelled graph

$$p \xrightarrow{a} q \iff V_p \subseteq W_a \;\&\; V_q \subseteq F_a V_p$$



A partition $\mathcal{V} = \{V_p : p \in B\}$ refines $\mathcal{W}$

$$V_p \subseteq W_a,\ V_q \cap F_a V_p \neq \emptyset \implies V_q \subseteq F_a V_p$$

$$p \xrightarrow{a} q \iff V_p \subseteq W_a \ \&\ V_q \subseteq F_a V_p$$

Let $p_0 \xrightarrow{u_0} p_1 \xrightarrow{u_1} \cdots \xrightarrow{u_{n-1}} p_n$ be a path.
$V_{p_i} \subseteq W_{u_i},\ F_{u_i}^{-1} V_{p_{i+1}} \subseteq V_{p_i}$.

$$\begin{aligned} F_{u_{[0,n)}}^{-1} V_{p_n} &\subseteq F_{u_{[0,n-1)}}^{-1} V_{p_{n-1}} \subseteq \cdots \subseteq F_{u_0}^{-1} V_{p_1} \subseteq V_{p_0}, \\ F_{u_{[0,n)}}^{-1} V_{p_n} &\subseteq F_{u_{[0,n-1)}}^{-1} W_{u_{n-1}} \cap \cdots \cap F_{u_0}^{-1} W_{u_1} \cap W_{u_0} \\ &\subseteq W_{u_{[0,n)}}, \end{aligned}$$

Thus $W_{u_{[0,n)}} \neq \emptyset$, so $u_{[0,n)} \in \mathcal{L}_{\mathcal{W}}$.

Conversely assume $W_u \neq \emptyset$

$$\exists p_0, \emptyset \neq V_{p_0} \cap W_u \subseteq V_{p_0} \cap W_{u_0} \implies V_{p_0} \subseteq W_{u_0}$$

$$\emptyset \neq V_{p_1} \cap F_{u_0}(V_{p_0} \cap W_u) \subseteq V_{p_1} \cap F_{u_0} V_{p_0} \cap W_{u_1},$$

$V_{p_1} \subseteq W_{u_1}$, $V_{p_1} \subseteq F_{u_0} V_{p_0}$, $V_{p_1} \cap F_{u_0} W_u \neq \emptyset$.
Let $V_{p_k} \cap F_{u_{[0,k)}} W_u \neq \emptyset$. $\exists p_{k+1}$

$$\begin{aligned} \emptyset &\neq V_{p_{k+1}} \cap F_{u_k}(V_{p_k} \cap F_{u_{[0,k)}} W_u) \\ &\subseteq V_{p_{k+1}} \cap F_{u_k} V_{p_k} \cap W_{u_{k+1}}, \end{aligned}$$

$V_{p_{k+1}} \subseteq W_{u_{k+1}}$, $V_{p_{k+1}} \subseteq F_{u_k} V_{p_k}$,
$V_{p_{k+1}} \cap F_{u_{[0,k+1)}} W_u \neq \emptyset$.

Let $\mathcal{S}_\mathcal{W}$ be sofic

$\mathcal{O}_u = \{v \in A^* : uv \in \mathcal{L}_\mathcal{W}\}$: the follower set of u

$\{\mathcal{O}_u : u \in \mathcal{L}_\mathcal{W}\}$ is a finite set.

$\mathcal{O}_u = \mathcal{O}_v$, $w \in A^*$, $W_{uw} \neq \emptyset \iff W_{vw} \neq \emptyset$,

$F_u W_u \cap W_w \neq \emptyset \iff F_v W_v \cap W_w \neq \emptyset$

$F_u W_u = F_v W_v$: $\{F_u W_u : u \in \mathcal{L}_\mathcal{W}\}$ is a finite set.

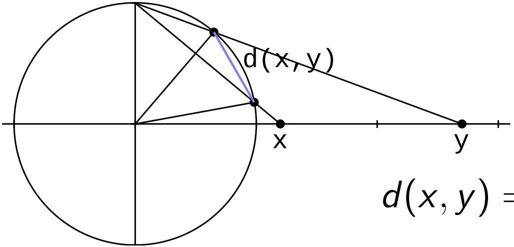
$\mathcal{V}$ is the refinement of $\mathcal{W} \cup \{F_u W_u : u \in \mathcal{L}_\mathcal{W}\}$.

$F_a(F_u W_u \cap W_a) = F_{ua}(W_u \cap F_u^{-1} W_a) = F_{ua} W_{ua}$

$F_a(W_a \setminus \overline{F_u W_u}) = F_a W_a \setminus \overline{F_a W_a \cap F_{au} W_u} =$

$F_a W_a \setminus \overline{F_{ua} W_{ua}}$

The chord metric on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$



$$d(x,y) = \frac{2|x-y|}{\sqrt{(x^2+1)(y^2+1)}}$$

Möbius transformations and the circle derivation:

$$F(x) = \frac{ax+b}{cx+d}, \ \det(F) = ad - bc > 0$$

$$F^\bullet(x) = \lim_{y \to x} \frac{d(F(y), F(x))}{d(y,x)}$$

$$= \frac{(ad-bc)(x^2+1)}{(ax+b)^2 + (cx+d)^2}$$

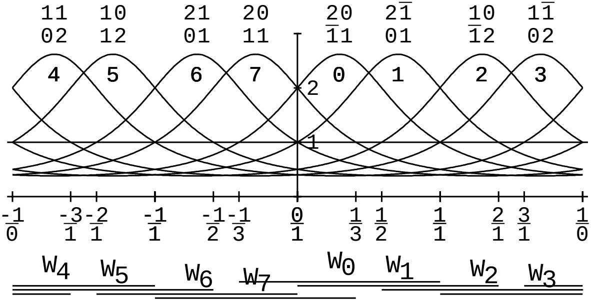
Möbius number systems

$F_a : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$: Möbius transformations.

If $W_a \subseteq \{x \in \overline{\mathbb{R}} : \; F_a^{\bullet}(x) > 1\}$, then
$\mathcal{W}$ is a generator, and
$\Phi : \mathcal{S}_{\mathcal{W}} \to \overline{\mathbb{R}}$ is a continuous surjection

$$\{\Phi(u)\} = \bigcap_{n \geq 0} \overline{W_{u_{[0,n)}}}, \; u \in \mathcal{S}_{\mathcal{W}}$$

The bimodular system



$$\det(F_k) = 2,\ ||F_k||^2 = 6,\ \mathrm{tr}(F_k) = 3$$

$$W_k = \{x \in \overline{\mathbb{R}} :\ F_k^{\bullet}(x) > 1\}$$

F_k is expansive on W_k, so $\mathcal{W}$ is a generator.

$V_0 = (0, \frac{1}{3}),\ V_1 = (\frac{1}{3}, \frac{1}{2}),\ V_2 = (\frac{1}{2}, 1), \ldots$

$\mathcal{S}_{\mathcal{W}}$ is sofic.

Arithmetical algorithms: compute a Möbius transformation

vertices (X, a), where
X is a Möbius transformation, $a \in A \cup \{\lambda\}$.

$$(X, a) \overset{\lambda/c}{\to} (X \circ F_c^{-1}, a) \text{ if } X(W_a) \subseteq W_c$$
$$(X, a) \overset{b/\lambda}{\to} (F_a \circ X, b)$$
$$(X, \lambda) \overset{a/\lambda}{\to} (X, a)$$

If $(M, \lambda) \overset{u/v}{\to}$, and $u \in \mathcal{S}_{\mathcal{W}}$, then $M \circ \Phi(u) = \Phi(v)$

$\det(X_{n+1}) = 2\det(X_n)$ or $\det(X_{n+1}) = \det(X_n)/2$,
$||X_n|| \leq C \cdot \det(X_n)$

Fractional bilinear functions

$$T(x, y) = \frac{T_0 xy + T_1 x + T_2 y + T_3}{T_4 xy + T_5 x + T_6 y + T_7}$$

If M is a Möbius transformation then
$M(T(x, y))$, $T(M(x), y)$, $T(x, M(y))$
are fractional bilinear functions.