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ANDRÁS HAJNAL

PIMS Distinguished Chair
University of Calgary
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On the Chromatic Number of Graphs and Set Systems

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András Hajnal

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Lecture 1

During this series of lectures, (I should have included this in the title) we are talking about infinite graphs and set systems, so this will be infinite combinatorics. This subject was initiated by Paul Erdős in the late 1940's.

I will try to show in these lectures how it becomes an important part of modern set theory, first serving as a test case for modern tools, but also influencing their developments.

In the first few of the lectures, I will pretend that I am talking about a joint work of István Juhász, Saharon Shelah and myself [23].

The actual highly technical result of this paper that appeared in the *Fundamenta* in 2000 will only be stated in the second or the third part of these lectures. Meanwhile I will introduce the main concepts and state—and sometimes prove—simple results about them.

In the spirit of Erdős, graph means simple graph , i. e., graphs with no loops and multiple edges.

Notation 1 (1) $G = (V, E)$ *is a graph if* $E \subset [V]^2 = \{a : a \subset V \wedge |a| = 2\}$.

(2) K_λ *denotes the complete graph of* λ *vertices for a cardinal* λ.

Remark. Often E alone is viewed as a graph, then we mean $V = \bigcup E$.

Definition 2 *Let* G *be a graph,* γ *an ordinal, then*

(1) $f : V \to \gamma$ *is a good coloring of* G *with* γ *colors if for every* $\{u, v\} \in E$, $f(u) \neq f(v)$.

(2) The chromatic number is defined as

$$\begin{aligned}\chi(G) &= \mathrm{Chr}(G) \\ &:= \min\{\kappa : \exists f : V \to \kappa \text{ such that } f \text{ is a good coloring of } G\}.\end{aligned}$$

We all know from the Four Color Theorem: if V is the set of faces of a planar map and $\{u, v\} \in E$ iff u, v have a common side, then $\mathrm{Chr}(G) \leq 4$.

One of the first questions of graph theory was what makes the chromatic number large. It would be nice to know that, and it is also very hard to find conditions for that. The next remark was the simplest observation:

Proposition 3 *If* $K_n \subset G$, *then* $\mathrm{Chr}(G) \geq n$.

Actually $K_n \subset G$ is not necessary to guarantee the large chromatic number.

Theorem 4 (Mycielski, [29]) *For every $n \in \omega$, there is a (finite) graph G_n such that $K_3 \not\subseteq G_n$ and $\mathrm{Chr}(G) \geq n$.*

Theorem 5 (Kőnig) *A graph has $\mathrm{Chr}(G) \leq 2$ iff it contains no odd cycle.*

In 1959, Erdős proved the following result, in a ground breaking paper on finite combinatorics.

Definition 6 *Given a graph G, the girth and odd girth are defined as*

$$\mathrm{Girth}(G) := \min\{r : G \textit{ contains a cycle of } r \textit{ edges }\},$$
$$\mathrm{Oddgirth}(G) := \min\{r : r \textit{ is odd and } G \textit{ contains a cycle of } r \textit{ edges}\}.$$

Theorem 7 (Erdős, [5]) *For all $n, r \in \omega$ there is a graph G such that*

$$\mathrm{Chr}(G) \geq n \quad \textit{and} \quad \mathrm{Girth}(G) \geq r.$$

This is one of the very first applications of the probabilistic method. Erdős always wanted to know the possible infinite generalizations.

Theorem 8 (Erdős, Rado 1960, [18]) *For every $\kappa \geq \omega$, there is a graph $G = (V, E)$ such that*

$$K_3 \not\subset G \quad \textit{and} \quad \mathrm{Chr}(G) = \kappa.$$

The above theorem is a generalization of Mycielski's Theorem. The graph they used was $\mathrm{Specker}(\kappa, 3) = E$, with vertex set $[\kappa]^3$, and for any two vertices $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$, where $\alpha_1 < \alpha_2 < \alpha_3$, $\beta_1 < \beta_2 < \beta_3$ and $\alpha_1 < \beta_1$, we have

$$\{\{\alpha_1, \alpha_2, \alpha_3\}, \{\beta_1, \beta_2, \beta_3\}\} \in E \quad \text{iff} \quad \alpha_2 < \beta_1 < \alpha_3 < \beta_2.$$

Now I want to make a short detour talking history. I met Erdős for the first time the spring of 1956 in Szeged, a small town in the southern part of Hungary, where I just finished writing my PhD. He lived abroad in the West but started to visit this year regularly to see his elderly mother and using this opportunity to work with a great many new people. Those days this was not a small matter to arrange and beside his worldwide fame this we could thank for the clever politics of some of his influential friends. We started to write long joint papers as soon as we met, and by 1960 we had quite a few to place in Hungarian journals. One of the editors suggested that why don't you rather write a book. (I suspect he was not very happy to read our manuscripts written in a rather informal style) Erdős seemed to like the idea, and I arrived one day to Mátraháza, a small place in the mountains, armed with periodicals I felt necessary for the preparation of this task. There was in this village a summerhouse for members of the Hungarian Academy where Erdős spent his vacation with his mother working simultaneously with many people. Rényi and Turán were usually there.

I gave one of the papers to Paul to read and for some reason I left the room. When I returned I found him reading excitedly, but not the paper I gave to him but the one next to it. That was where we found property **B**, which was defined by Miller in this remarkable paper written in 1937.

Definition 9 *Let $\mathcal{F}$ be a family of sets, $\mathcal{F}$ has property* **B** *if there is a set $B \subseteq \cup\mathcal{F}$, such that for every $F \in \mathcal{F}$, $F \cap B$, $F - B$ are both nonempty.*

Felix Bernstein proved that the set of perfect subsets of $\mathbb{R}$ has property **B**, i. e., there is a set $B \subseteq \mathbb{R}$ such that neither B nor its complement contains a perfect subset. Miller invented the name property **B** in honor of Bernstein.

Theorem 10 (Miller 1937, [28]) *Let $n \in \omega$. Assume $\mathcal{F}$ is a family of sets that are of size ω, and $|F_1 \cap F_2| \leq n$ for $F_1 \neq F_2 \in \mathcal{F}$. Then $\mathcal{F}$ has property* **B**.

Erdős liked the subject and started to conjecture generalizations even before we properly read the paper. Anyway reading meant reading the theorems and making the proofs. This time this did not quite work. I had to read the proof.

First we proved that under this condition, $\mathcal{F}$ has even a stronger property.

Definition 11 *Let $\mathcal{F}$ be a family of sets, $\mathcal{F}$ has property $\mathbf{B}(\gamma)$ if there is a set B such that*

$$0 < |B \cap \mathcal{F}| < \gamma, \quad \textit{for all } F \in \mathcal{F}.$$

Now we can state a theorem as follows.

Theorem 12 (Erdős, Hajnal 1961, [9]) *If $\mathcal{F}$ consists of sets of size ω, and if $|F_1 \cap F_2| < n$ for some $n < \omega$, then $\mathcal{F}$ has property $\mathbf{B}(\omega)$.*

Remark. In the theorem above in general we can not have property $\mathbf{B}(k)$ for $k < \omega$ but I will not discuss this.

The real generalizations are when the sets are assumed to be larger. Before stating them I will outline a proof of Miller's theorem. But first let me rephrase property **B** generalizing the concept of chromatic number and revealing the connection with the subject discussed before.

Assuming $|F| \geq 2$ for each $|F| \in |\mathcal{F}|$, what does it mean that $\mathcal{F}$ has the property **B**? Assume that $\mathcal{F}$ consists of subsets of S, define $f(x)$ for $x \in S$

$$f(x) = \begin{cases} 0, & \text{if } x \in B, \\ 1, & \text{if } x \notin B. \end{cases}$$

Then property **B** means exactly that there is a coloring of the vertices such that for any $F \in \mathcal{F}$, f is not constant on F.

Definition 13 *Let $\mathcal{F}$ be any set system, such that $|F| \geq 2$, for each $F \in \mathcal{F}$. Then we define*

$$\mathrm{Chr}(\mathcal{F}) := \min\{\kappa : \textit{ there is } f : \cup\mathcal{F} \to \kappa, \textit{ such that } f \restriction F \textit{ is not constant, for every } F \in \mathcal{F}\}.$$

Therefore, $\mathcal{F}$ has property **B** iff $\mathrm{Chr}(\mathcal{F}) = 2$. This explains why we have similar phenomena in the theorems concerning the two concepts. And indeed the theorem below was proved with method to be discussed and generalized in the next lectures.

Theorem 14 (Erdős, Hajnal 1966, [10]) *If* $\mathrm{Chr}(G) > \omega$*, then* $C_4 \subseteq G$*, where* C_4 *denotes the cycle of length 4.*

More generally,

Theorem 15 *If* $\mathrm{Chr}(G) > \omega$*, then* $K_{l,\omega_1} \subseteq G$ *for every* $l < \omega$*.*

As a consequence, a G of chromatic number greater than ω must contain every finite bipartite graph and as a corollary of this every even cycle. This leaves open the problem of short odd cycles only, starting with the pentagon.

As soon as we discovered the theorem we could prove the following result.

Theorem 16 (Erdős Hajnal 1966, [11]) *For every* $\kappa \geq \omega$ *and every* r*, there is a graph* G*, with* $\mathrm{Chr}(G) = |G| = \kappa$*, such that*

$$\mathrm{Oddgirth}(G) > r.$$

I finish this lecture with a theorem, a corollary of which leads to an interesting unsolved problem.

Theorem 17 (Erdős, Hajnal, Shelah 1972 [17]) *If* $\mathrm{Chr}(G) > \omega$*, then for some* r_0*,* G *contains all cycles* C_{2r+1} *for any* $r > r_0$*.*

Corollary 18 *If* G_1*,* G_2 *are two graphs each has* $\mathrm{Chr}(G_i) \geq \omega$*,* $i = 1, 2$*. Then there exists* H *such that*

$$\mathrm{Chr}(H) = 3 \quad \textit{and} \quad H \subseteq G_i, \ \textit{for } i < 2.$$

Problem 19 (Erdős) *Can* 3 *be replaced by* 4 *in the above corollary?*

One last remark: I learned from Erdős to always look at the simplest case of the unsolved problem we were thinking about. This is often usefull but can sometimes be misleading. Had we tried to generalize the finite theorem not only for quadrilaterals but also for pentagons we would have discovered the truth much faster.

Lecture 2

In the last lecture, I promised a proof of Miller's Theorem. Now let's recall this theorem as follows.

Theorem 20 (Miller 1937, [28]) *Let $n \in \omega$. Assume $\mathcal{F}$ is a family of sets that are of size ω, and $|F_1 \cap F_2| \leq n$ for $F_1 \neq F_2 \in \mathcal{F}$. Then $\mathcal{F}$ has property* $\mathbf{B}$.

In fact, Erdős and Hajnal proved in 1961 that under this condition, $\mathcal{F}$ has even stronger property $\mathbf{B}(\omega)$.

Proof of Theorem 20. We will use cardinal induction. Let $|\mathcal{F}| = \kappa$. We prove the statement by induction on κ.

Case(i)

We can enumerate $\mathcal{F} = \{F_i : i < \omega\}$, and choose x_i by induction on i so that $x_i \in F_i$. Assume we have chosen $x_j \in F_j$, for $j < i$, in such a way that for $X_i = \{x_j : j < i\}$

$$|X_i \cap F_l| \leq n + 1, \quad \text{for all} \quad l < \omega.$$

(Note that x_j is not necessarily one-to-one) If $X_i \cap F_i \neq \emptyset$, we choose $x_i = x_j$ for the x_j showing this.

Assume $X_i \cap F_i = \emptyset$. There are infinitely many possibilities for $x_i \in F_i$. If we can not choose $x_i \in F_i$ satisfying the above equation, then for each $x \in F_i$ there are, $T(x) \in [X_i]^{n+1}$ and $l(x) \in \omega, l(x) \neq i$ such that $T(x) \cup \{x\} \subseteq F_{l(x)}$ and $|T(x)| = n + 1$,. Then $T(x) = T \subseteq F_{l(x)}$ for infinitely many $x \in F_i$.Then $l(x) = l$ for all these x's. Thus $|F_l \cap F_i| = \omega$, contradiction.

Assume now $\kappa > \omega$ and the statement is true for $|\mathcal{F}| < \kappa$. We need to see that the result is true for $|\mathcal{F}| = \kappa$ Instead of writing down the original proof, we now learn a basic method of modern set theory, the method of elementary submodels and elementary chains, which evolved 40–50 years later in the works of Shelah. This of course is a waste of our shot for the special case, but we hope that it will help to understand the general method.

First of all, we need some concepts of logic. Recall that a structure is a pair $\mathcal{A} = \langle A, R \rangle$, where $A \neq \emptyset$ and R is a relation over A. We will only consider structures

$$\langle A, \epsilon \rangle \quad \text{or} \quad \langle A, \epsilon \restriction A \rangle,$$

where $\epsilon \restriction A := \{\langle x, y \rangle : x, y \in A \wedge x \in y\}$.

Definition 21 *We say that the structure $\mathcal{A}$is an elementary substructure of the structure $\mathcal{B}, \mathcal{A} \prec \mathcal{B}$ in notation, if $A \subseteq B$, and for all formalas $\Psi(x_1, \ldots, x_n)$ with free variables $x_1 \ldots x_n$ and $\forall a_1, \ldots, a_n \in A$, we have*

$$\mathcal{A} \models \Psi(a_1, \ldots, a_n) \quad (= \Psi_{\mathcal{A}}(a_1, \ldots, a_n))$$

iff

$$\mathcal{B} \models \Psi(a_1, \ldots, a_n) \quad (= \Psi_{\mathcal{B}}(a_1, \ldots, a_n))$$

We need the well-known

Theorem 22 (Lőwenheim, Skolem) *For all $C \subseteq B$, $\lambda \geq \omega$, $|C| \leq \lambda$, there is an A, such that $C \subset A$, $|A| = \lambda$, and*

$$\mathcal{A} \prec \mathcal{B}.$$

This says that we can extend a given small size elementary submodel of a large model to a larger size. But—as it is well known—we can not insist on elementary submodels of $\langle V, \in \rangle$, where V is the class of all sets. That is why we need some big models similar to $\langle V, \in \rangle$. Big is meant "big relative to some fixed cardinal".

Definition 23 *For a regular cardinal λ, $H(\lambda)$ is the set of sets hereditarily of cardinality less that λ.*

Remark. If $x \in H(\lambda)$ then $|x| < \lambda$, and if $y \in x \in H(\lambda)$ then $|y| < \lambda$.

We can prove the existence of $H(\lambda)$ as follows: Let H_α be defined by recursion on α as follows,

$$\begin{aligned} &H_0 = \emptyset, \\ &H_{\alpha+1} = H_\alpha \cup [H_\alpha]^{<\lambda}, \\ &H_\beta = \bigcup_{\alpha<\beta} H_\alpha, \quad \text{for} \beta \text{ limit}, \end{aligned}$$

and then,

$$H(\lambda) = H_\lambda.$$

Remark. $H(\lambda)$ is very similar to $\langle V, \in \rangle$. $\langle H(\lambda), \in \rangle$ for a regular λ is a model of all axioms of ZFC without the powerset axiom.

In the following we will use $\lambda = \kappa^+$, the successor of κ. Also we will denote $\langle N, \in \rangle$ shortly by N, a jargon generally accepted in set theory.

Now let us present elementary chain method in case of Miller's theorem. First we will define a sequence $N_\alpha : \alpha < \kappa$ by recursion on α.

(i) $N_0 = \emptyset$, $N_1 \prec N_2 \prec \cdots \prec N_\alpha \prec \cdots \prec H(\kappa^+)$,

(ii) $\mathcal{F}, \kappa \in N_1$, $N_\alpha \in N_{\alpha+1}$,

(iii) $|N_\alpha| \leq |\alpha| + \omega$,

(iv) $N_\alpha = \bigcup\{N_\beta : \beta < \alpha\}$ for α a limit ordinal.

Theorem 22, makes this definition possible. Also we may assume $\bigcup \mathcal{F} = \kappa$, and as a consequence of this $F \subset \kappa$ for $F \in \mathcal{F}$.

Setting $S_\alpha = N_\alpha \cap \kappa$, $R_\alpha = S_{\alpha+1} - S_\alpha$, we have

$$\kappa = \bigcup_{\alpha<\kappa} R_\alpha.$$

where the sets R_α are pairwise disjoint.

For $F \in \mathcal{F}$, let

$$\alpha(F) := \min\{\alpha : |F \cap S_{\alpha+1}| \geq n+1\},$$
$$\mathcal{F}_\alpha := \{F \in \mathcal{F} : \alpha(F) = \alpha\},$$

(1) If $F \in \mathcal{F}_\alpha$, then $|F \cap S_\alpha| \leq n$ by definition.

We now prove the following statement.

(2) If $F \in \mathcal{F}_\alpha$, then $F \in N_{\alpha+1}$ and $F \subseteq S_{\alpha+1}$.

Indeed, for $X \in [\kappa]^{n+1}$, let

$$\Phi(X) = \begin{cases} F, & \text{if F is the unique element of } \mathcal{F} \text{ such that } X \subseteq F, \\ X, & \text{otherwise.} \end{cases}$$

As this is an elementary definition, $X \in N_\alpha$ implies $\Phi(X) \in N_\alpha$. If $F \in \mathcal{F}_\alpha$ let $X \in [S_{\alpha+1} \cap F]^{n+1}$. Then $X \in N_\alpha$, $\Phi(X) \in N_\alpha$ Therefore $F \in \mathcal{F}_\alpha$ implies $F \in N_{\alpha+1}$.

To see the second claim of (2) we notice that there is $\psi : \omega \to F$ for $F \in \mathcal{F}_\alpha$ which is onto, and clearly $\psi \in H(\kappa^+)$. It is also obvious that then

$$H(\kappa^+) \models \exists\psi(\psi : \omega \to F \wedge Ran(\psi) = F)$$

and by elementarity, there is a $\psi \in N_{\alpha+1}$ satisfying the same formula in $N_{\alpha+1}$. Then

$\forall \xi \in F, \quad \exists n \in \omega, \psi(n) = \xi$ and this implies the second claim.

By (2) we know that $|\mathcal{F}_\alpha| \leq |N_{\alpha+1}| < \kappa$, and the same holds for

$$\widetilde{F_\alpha} = \{F \cap R_\alpha : F \in \mathcal{F}\alpha\}.$$

Then by induction hypothesis, there is $B_\alpha \subseteq R_\alpha$, such that

$$0 < |B_\alpha \cap F| < \omega, \quad \text{for} \quad F \in \mathcal{F}_\alpha.$$

Let $B = \bigcup_{\alpha<\kappa} B_\alpha$, then $0 < |B \cap F| \leq n + 1 + |B_\alpha \cap F| < \omega$ for $F \in F_\alpha$. □

Remark. Note that this implies that if $|\mathcal{F}| \le \omega_k$ then $|F \cap B| \le (k+1)n + 1$, and this is known to be best possible assuming GCH (Erdős, Hajnal 1961, [9]).

Miller of course did not define elementary submodels. He defined the sets S_α to be closed with respect to the specific Skolem functions needed to verify (1) and (2).

In 1961, we were interested in how to generalize Miller's result for $|F| = \omega_1$ instead of ω. Then $\mathcal{F}$ consists of sets of size ω_1, the assumption $|F_1 \cap F_2| \le n$ is then too much, and $|F_1 \cap F_2| \le \omega_1$ is not enough. We finally found that $|F_1 \cap F_2| \le \omega$ is the right assumption. We state the simplest instance of the generalization first.

Theorem 24 (Erdős, Hajnal 1961, [9]) *Assume $2^{\aleph_0} = \aleph_1$, $\mathcal{F}$ consists of sets of size of ω_1 and $\mathcal{F}$ is strongly almost disjoint, i. e. $|F_0 \cap F_1| < \omega$ for any $F_0, F_1 \in \mathcal{F}$. Then $\mathcal{F}$ has property $\mathbf{B}(\omega_1)$, provided $|\mathcal{F}| \le \aleph_\omega$.*

Remark. In the theorem above, we can have $\mathbf{B}(\omega_1)$, but not $\mathbf{B}(\omega)$.

We proposed the problem: what happens with the above theorem for $|\mathcal{F}| = \aleph_\omega$

Taking the cue from Richard Rado who invented the ordinary partition symbol we also introduced a general symbol involving many parameters to formulate our results. I give this definition now.

Definition 25 *The notation*

$$M(\kappa, \lambda, \mu) \to \mathbf{B} \quad (\textit{resp. } \mathbf{B}(\sigma))$$

means that if $\mathcal{F} \subseteq [\kappa]^\lambda$, $|\mathcal{F}| = \kappa$, $\mathcal{F}$ is μ–disjoint (i.e. $|F_0 \cap F_1| < \mu$ for $F_0 \ne F_1 \in \mathcal{F}$) then $\mathcal{F}$ has property $\mathbf{B}$ (resp. $\mathbf{B}(\sigma)$).

And the notation

$$M(\kappa, \lambda, \mu) \nrightarrow \mathbf{B} \quad (\textit{resp. } \mathbf{B}(\sigma))$$

means the negation of the statement.

Many people dislike this habit but I still find it usefull if used with good taste end moderation. However I will not burden this section with stating the general result. This will be done in the next lecture. In 1961 we did not quite understand the strength of the method we rediscovered. We only proved the graph theorem mentioned a few years later. I will talk about this proof first in the next section.

Lecture 3

First of all, let us recall the elementary chain method.

Given κ, we can choose a sequence $N_\alpha : \alpha \prec \kappa$,

(i) $N_0 = \emptyset$, $N_1 \prec N_2 \prec \cdots < N_\alpha \prec \cdots \prec H(\kappa^+)$,

(ii) $\cdots \in N_1$, $N_\alpha \in N_{\alpha+1}$,

(iii) $|N_\alpha| \leq |\alpha| + \omega$,

(iv) $N_\alpha = \bigcup N_\beta$, $\beta < \alpha$ if α is a limit ordinal.

Set $S_\alpha = N_\alpha \cap \kappa$, $R_\alpha = S_{\alpha+1} - S_\alpha$, $\alpha < \kappa$. R_α are pairwise disjoint. In the sequence above, we can put anything we want into ... in (ii).

Before stating the graph theorem, we introduce a new concept.

Definition 26 *Let $G = (V, E)$ be a graph, for a well-ordering $\prec$ of V, for $x \in V$, define*

$$G_\prec(x) = \{y \in V : y \prec x \quad \textit{and } \{x, y\} \in G\}.$$

Definition 27

$$\mathrm{Col}(G) := \min\{\kappa : \textit{there is a well ordering } \prec \textit{ of } V, \textit{ such that } |G_\prec(x)| < \kappa \textit{ for all } x \in V\}.$$

Lemma 28 *If $\prec$ satisfies $|G_\prec(x)| < \kappa$ then we can define a good coloring $f(x)$ by recursion on x.*

Corollary 29 $\mathrm{Col}(G) \geq \mathrm{Chr}(G)$.

The next theorem is not this straightforward, but I omit the proof.

Theorem 30 *If $\mathrm{Col}(G) = \kappa$, then there is a well-ordering of type κ showing this.*

Example. Let $G = K_{\omega,\omega_1}$. Then $\mathrm{Chr}(G) = 2$, and by the above theorem, $\mathrm{Col}(G) > \omega$.

Now we can state a generalization of the Erdős, Hajnal graph theorem.

Theorem 31 *If* $K_{l,\omega_1} \not\subseteq G$ *for some* $l < \omega$ *then* $\mathrm{Col}(G) \leq \omega$.

Proof. We use cardinal induction. It is trivial for $\kappa \leq \omega$.

Assume that it is true for $\lambda < \kappa$ for some $\kappa > \omega$. Let $|G| = \kappa$, we may assume that $V = \kappa$. Assume for contradiction that $K_{l,\omega_1} \not\subseteq G$ for some $l < \omega$. Put $G, \kappa \in N_1$.

For $X \in [\kappa]^l$, let

$$\Phi(X) := \{y \in V : \forall u \in X, \{u, y\} \in E\}.$$

Thus $|\Phi(X)| \leq \omega$, Φ is defined in $(N_\alpha, \in)$.

Therefore, just like in the proof given in Lecture 2, if $X \in [S_\alpha]^l$, then $\Phi(X) \subseteq S_\alpha$. As a corollary, if $u \in R_\alpha$, then $|G(u) \cap S_\alpha| < l$, where

$$G(u) := \{v : \{u, v\} \in E\}.$$

By the induction hypothesis there is a well-ordering $\prec_\alpha$ of R_α showing that G has coloring number $\leq \omega$ on R_α. The lexicographic sum $\prec$ of the $\prec_\alpha$'s is a well-ordering of κ that shows $\mathrm{Col}(G) \leq \omega$. □

Let us go back to the room where Erdős started conjecturing: Let G be assumed not to contain K_{ω,ω_2} and try to prove $\mathrm{Col}(G) \leq \omega_1$.

Looking back to the last proof, $l < \omega$ will have to be replaced by ω. To have some hope S_α has to be closed with respect to countable subsets, We must have $[N_\alpha]^\omega \subseteq N_\alpha$ For this we will have to assume the Continuum Hypothesis, CH, and for the general theorem the Generalized Continuum Hypothesis, GCH. As GCH itself is independent of the axioms of set theory, the reader (audience)has to trust our judgement that it is more interesting to investigate the problem under GCH then without it. Assuming GCH and have many theorems was certainly our preference. Following the old proofs and using the elementary chain method we will get up to $\kappa \leq \aleph_\omega$

Theorem 32 *Assume GCH,*

a) *If* $|G| \leq \kappa \leq \aleph_\omega$, $K_{\omega,\omega_2} \not\subseteq G$, *then* $\mathrm{Col}(G) \leq \omega$.

b) $\mathcal{F}$ *consists of set of size* ω_1, $|\mathcal{F}| \leq \aleph_\omega$, $|F_0 \cap F_1| < \omega$ *for* $F_0 \neq F_1$ *then the family* $\mathcal{F}$ *has property* $\mathbf{B}(\omega_1)$.

c) $\mathcal{F}$ *consists of set of size* $\aleph_\omega$, $|\mathcal{F}| \leq \aleph_\omega$, $|F_0 \cap F_1| < \omega$ *for* $F_0 \neq F_1$ *then the family* $\mathcal{F}$ *has property* $\mathbf{B}(\omega_1)$.

Remark. The case $|\mathcal{F}| = \kappa = \aleph_{\omega+1}$ of all three statements is independent of ZFC (relative to some large cardinal). This is proved for a) and b) in the first paper by Hajnal, Juhász and Shelah in 1986 [22] and for c) in 2000 [23] by the same authors.

The consistency with GCH is the "easy" one. Assuming say the axiom of constructibility $V = L$ all three are true for all κ, in each instance one can use the corresponding instance of Jensen's $\Box_\kappa$ principle, or some consequence of it. This is really due to no one, it was just realized by many people. For the definitions of different large cardinals, $V = L$ and $\Box_\kappa$ we refer to [24].

An interesting curiosity: We already realized in 1960 that the " problems for $\aleph_{\omega+1}$" are very interesting and devised a statement that would give a positive solution.

For each $\alpha < \aleph_{\omega+1}$ there is a partition $\{S_n^\alpha : n < \omega\}$ of α such that for each $\alpha < \aleph_{\omega+1}$

$$|S_n^\alpha| = \aleph_n$$

and for $cf(\alpha) = \omega_1$ there is a sequence $\alpha_\nu \nearrow \alpha$, $\nu < \omega_1$ such that for each $n < \omega$ $\langle S_n^{\alpha_\nu} : \nu < \omega_1\rangle$ is increasing in ν.

This easily follows from $\Box_{\aleph_\omega}$, but of course in 1960 we had no idea how to prove it. It took the insight of Jensen, to show $\Box_{\aleph_\omega}$ from the constructibility axiom. Later it was realized that they can be forced in with easy forcings. After that the problem was to make them false.
This requires large cardinals. By the results of [22], the negations of a), b) are consistent relative to a supercompact cardinal.

By the results of [23], the negation of c) is consistent relative to an even larger cardinal.

Here is a general form of the Theorem 32 stated on page 10,using the definition of

$$M(\kappa, \lambda, \mu) \to \mathbf{B} \quad (\text{resp. } \mathbf{B}(\sigma))$$

given in Lecture 2.

Theorem 33 *Assume GCH, ρ is regular and $\lambda \le \kappa$. Then*

a) $M(\kappa, \lambda, \rho) \to \mathbf{B}(\rho^+)$, for $\rho \le \rho^{(+\rho)}$;

b) $M(\kappa, \lambda, \rho) \to \mathbf{B}(\rho^+ +)$, for all ρ.

Here $\rho^{+\sigma}$ denotes the σ-th successor of the cardinal ρ. If in addition, $\Box_\kappa$ holds for every κ, e. g. if $V = L$ is true, then in a) we can drop the assumption $\rho \le \rho^{(+\rho)}$. In HJS we proved that the theorem is best possible.

Here is the precise form.

Theorem 34 *Assume GCH and ρ is regular.*

a) If there is a supercompact cardinal above ρ, then there is a ρ-closed forcing notion P such that in V^P, GCH is true and

$$M(\rho^{(+\rho+1)}, \rho^+, \rho) \nrightarrow \mathbf{B} \quad \textit{holds.}$$

b) If there is a 2-huge cardinal above, then

$$M(\rho^{+(\rho+1)}, \lambda, \rho) \to \mathbf{B}(\rho^+) \quad \textit{holds for all } \lambda \le \rho^{(+(\rho+1))}.$$

This is in [23]. Instead of 2-huge we use the following weaker property. There is an elementary embedding $\gamma : V \to M$ with $\text{crit}(\gamma) = \kappa$, $\gamma(\kappa) = \lambda$ and $M^{\lambda+(\rho+3)} \subseteq M$.

The necessity of using large cardinals is proved in [1, 23, 25].

Lecture 4

In this lecture, I am going to talk about compactness properties of the chromatic number.

Theorem 35 (de Bruijn, Erdős 1951) *Assume* $k < \omega$*, and* $\mathrm{Chr}(G') \leq k$ *for every finite* $G' \subseteq G$*. Then* $\mathrm{Chr}(G) \leq k$*.*

This we call the compactness of the chromatic number in ω. The proof follows from both, Gődel's compactness theorem and Tychonov's product theorem.

Can ω be replaced by ω_1?

Let us consider the simplest case. Assume $|G| = (|V|) = \omega_2$. Assume $\mathrm{Chr}(G') \leq \omega$ for every $G' \subseteq G$, $|G'| < \omega_2$.

Is $\mathrm{Chr}(G') \leq \omega$ or $\mathrm{Chr}(G') \leq \omega_1$?

We started to ask this in the early 60's. Finally we solved the first question in the negative assuming CH in [12].

Theorem 36 *For every* $\kappa \geq \omega$*, there is a graph* G *on* $V = (2^\kappa)^+$ *such that the* $\mathrm{Chr}(G) \geq \kappa^+$ *and for every* $G' \subseteq G$ *with* $|G'| \leq 2^\kappa$*,* $\mathrm{Chr}(G') \leq \kappa$*.*

Here is the proof for $2^\omega = \omega_1$, $\kappa^+ = \omega_2$.

Proof. Define a graph on $V = [\omega_2]^2$, $|V| = \omega_2$,

$$E = \mathrm{Shift}(\omega_2, 2) = \{\{\{\alpha, \beta\}, \{\beta, \gamma\}\} : \alpha < \beta < \gamma < \omega_2\}.$$

First see $\mathrm{Chr}(G) \geq \omega_1$.

Assume $f : [\omega_2]^2 \to \omega$ is a coloring of the vertex set with ω colors. By the Erdős-Rado theorem $2^\omega \to (\omega_1)^2_\omega$ there is a monochromatic triangle $\{\alpha, \beta, \gamma\}$ $\alpha < \beta < \gamma < \omega$. Then $\{\{\alpha, \beta\}, \{\beta, \gamma\}\} \in E$ and $f(\{\alpha, \beta\}) = f(\{\beta, \gamma\})$, f is not a good coloring.

Note that $\mathrm{Shift}(\kappa, 2)$ does not contain a triangle. If $n \to (3)^2_\kappa$ for n, κ, then $\mathrm{Shift}(n, 2)$ is a graph with chromatic number no less than κ containing no triangle (This gives -as promised-a proof of the Mycielski's result mentioned in Lecture 1).

To finish the proof we must see that

$$\mathrm{Chr}(\mathrm{Shift}(\alpha, 2)) \leq \omega$$

for $|\alpha| \leq 2^\omega$. It is easy to see that one can choose a sequence

$$\{A_\beta : \beta < \alpha\}$$

of subsets of ω such that A_β is not a subset of A_γ for $\beta \neq \gamma < \alpha$. We can define a partition proving this by :

$$f(\{\beta, \gamma\}) = \min(A_\beta \setminus A_\gamma)$$

for $\beta < \gamma$. □

It can be seen directly that

$$\mathrm{Chr}(\mathrm{Shift}(2^n + 1, 2)) \geq n + 1, \quad \text{while}$$
$$\mathrm{Chr}(\mathrm{Shift}(2^n, 2)) \leq n, \quad \text{for } n \in \omega.$$

The next result appeared eighteen years later. Baumgartner in [2] gave a forcing construction for a graph G with $\mathrm{Chr}(G) = |G| = \omega_2$ all whose subgraphs of size ω_1 have chromatic number no more than ω.

This says that "the chromatic number may jump 2". Six years later, Shelah proved the following theorem.

Theorem 37 (Shelah,1990) *If $V = L$ and κ is not weakly compact, i. e. $\kappa \not\to (\kappa)^2_2$ then there is a G with $\mathrm{Chr}(G) = |G| = \kappa$ such that for all $G' \subseteq G$ and $|G'| < \kappa$, we have $\mathrm{Chr}(G') \leq \omega$.*

Remark. Therefore there may be "arbitrarily large"jumps.

On the other hand, Foreman and Laver proved it consistent with GCH relative to a large cardinal in [19] in 1988 that every graph on ω_2 with $\mathrm{Chr}(G) = \omega_2$ contains a subgraph G' of chromatic number and size ω_1. Thus the second problem we stated at the beginning of the lecture is independent. We will come back to this later.

The following is a usual definition of the product of two graphs. Assume $G_i = \langle V_i, E_i \rangle, i < 2$ are graphs. Define $G_0 \times G_1$ as follows:

$$V(G_0 \times G_1) = V_0 \times V_1, \quad \{(x_0, x_1), (y_0, y_1)\} \in E(G_0 \times G_1)$$

iff $\{x_i, y_i\} \in E_i$ for $i < 2$.

Hedetniemi conjectured that for $n \in \omega$, $n \geq 2$ the product of two n chromatic graphs is n chromatic.

This was proved for $n = 3, 4$ by El-Zahar and Sauer, and is unsolved for $4 < n < \omega$.

During one of my visits in Calgary, I proved the following result.

Theorem 38 *The conjecture is true for $n = \aleph_0$ and false for $n = \kappa^+$ provided $\kappa \geq \omega$.*

It occured to me to ask if it is possible that the product of two ω_2-chromatic graphs is at most ω-chromatic.

Here is the connection: assume that G_i are ω_2 chromatic for $i < 2$ and $\mathrm{Chr}(G_0 \times G_1) \leq \omega$. Then each G_i is an example of an ω_2-chromatic graph all whose subgraphs of size ω_1 are at most ω-chromatic.

Proof. Assume for contradiction that, say, there is a $U \subseteq V_0$ such that $G[U]$ is ω_1 chromatic, and $|U| = \omega_1$. Assume $f : V_0 \times V_1$ shows that $G_0 \times G_1$ is ω-chromatic. Then for each $y \in V_1$ there are $x_1(y) \neq x_2(y) \in U$ such that $f(\langle x_1(y), y\rangle) = f(\langle x_2(y), y\rangle)$.

As $|U \times V| = \omega_1$, this partitions V_2 to ω_1 calsses each of which is independent for G_2 in contradiction and the fact that G_2 is ω_2-chromatic. □

Now, we have the following theorem.

Theorem 39 (Soukup, [33]) *It is consistent with GCH that there are two ω_2-chromatic graphs of size ω_2 whose product is ω-chromatic.*

This is a generalization of Baumgartner's theorem mentioned. But no one succeeded in the generalization of Shelah's result.

Problem 40 *Is it consistent with GCH that there are graphs G_i with $\mathrm{Chr}(G_i) = \omega_3$ for $i < 2$ so that $\mathrm{Chr}(G_0 \times G_1) \geq \omega_3$?*

Turning back to the Foreman-Laver result, we defined with Erdős a graph, say $\mathrm{Eh}(\omega_2, \omega) = Eh$, with $V = {}^{\omega_2}\omega$, $\{f, g\} \in E$ iff f, g are eventually different i. e. for some $\alpha < \omega_2$, $f(\beta) \neq g(\beta)$ for $\beta > \alpha$.

It is obvious that every subgraph of size at most ω_1 of this graph Eh is ω-chromatic, moreover every such graph of cardinality $\leq \omega_2$ can be embedded to it.

Proof of this fact: Assume $G = \langle \omega_2, E\rangle$ and all subgraphs of size $\leq \omega_1$ of it are ω-chromatic. Then for each $\alpha < \omega_2$ there is an $f_\alpha : \alpha \to \omega$ showing this.

Now we define the embedding $\Phi : G \longrightarrow Eh$ by defining $\Phi(\beta)$ for $\beta < \omega_2$, $\Phi(\beta) \in {}^{\omega_2}\omega$ by

$$\Phi(\beta)(\alpha) = \begin{cases} f_\alpha(\beta), & \text{for } \beta < \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

□

It follows from the Erdős-Hajnal theorem mentioned that

$$2^\omega = \omega_1 \quad \text{implies} \quad \mathrm{Chr}(Eh) \geq \omega_1.$$

Foreman strengthened his earlier mentioned result with Laver by proving it consistent relative to a large cardinal that $\mathrm{Chr}(Eh) \leq \omega_1$.

He did this by proving consistent that there is an ω_1 complete, ω_1-dense set ideal I in $\mathcal{P}(\omega_2)$. (This implies easily that $\mathrm{Chr}(Eh) \leq \omega_1$).

Remember: I is ω_1-complete if

$$I' \subseteq I \wedge |I'| \leq \omega \Longrightarrow \bigcap I' \in I.$$

I is ω_1-dense if there is $\mathcal{F} \subseteq I^+$, $|\mathcal{F}| \leq \omega_1$ such that for all $A \in I^+$ there is $B \in \mathcal{F}$ with $B - A \in I$. Note that I^+ is the complement of I.

Finally Todorčević [36] proved in ZFC that

$$\mathrm{Chr}(Eh) \geq \omega_1.$$

This gives us the unexpected

Corollary 41 (ZFC) *There exists a graph G of chromatic number greater than ω that has size greater than ω_1 all whose subgraphs of size at most ω_1 have chromatic number at most ω.*

But it does not answer our first problem as stated, since

$$|Eh| = \omega^{\omega_2} \geq \omega_3.$$

Lecture 5

In this section, I will focus on sets of finite subgraphs of uncountable chromatic graphs.

In 1979, we wrote a triple paper with Erdős and Szemerédi [15] in which we stated a few problems that were all mixtures of finite combinatorics and set theory which all involved the subject just mentioned.

First I will speak about one of them, I may come back to some other if I will have time.

We first need a generalization of the graph $\mathrm{Shift}(\kappa, 2)$.

Definition 42 *For α an ordinal, $r \in \omega$, $r \geq 2$, $\mathrm{Shift}(\alpha, r)$ is a graph with vertex set $V = [\alpha]^r$, edge set $E = \{\{\alpha_0, \ldots, \alpha_{r-1}\}, \{\alpha_1, \ldots, \alpha_r\}\}$, where $\alpha_1 < \cdots < \alpha_r < \alpha$.*

Fact 1. $\mathrm{Shift}(\alpha, r) \not\supset C_{2i+1}, \quad 1 \leq i < r$.

Fact 2. $\mathrm{Chr}(\mathrm{Shift}(\alpha, r)) \geq \kappa^+ \quad \text{iff} \quad |\alpha| \geq \exp_{r-1}(\kappa)^+$.

The proof of the first statement is an exercise in finite combinatorics. We proved the case $r = 2$ in the last lecture, the proof of this statement uses the same ideas.

Definition 43 *For a graph G of $\mathrm{Chr}(G) \geq \omega$ define $f_G : \omega \to \omega$*

$$f_G(n) := \min\{|A| : A| \subseteq V \wedge \mathrm{Chr}(G[A]) \geq n\}$$

for $n \in \omega$.

Corollary 44

$$f_{\mathrm{Shift}(\alpha,r)}(n) \geq \exp_{r-1}(n),$$

for $r \geq 2$, $n \geq n_r$, $\alpha \geq \omega$.

This motivated us to ask the problem: is it true that for every $f : \omega \to \omega$, there is a graph G with

$$f_G(n) \geq f(n), \quad \text{for } n \geq 3.$$

The following results are all from a forthcoming paper [26] of P. Komjáth and S. Shelah.

Theorem 45 (Shelah, [26]) *For every model of ZFC+$\diamondsuit$+GCH, there is a forcing extension P of cardinality ω_1 in which the answer to the problem is* yes*, GCH and $\diamondsuit$ remain true.*

Remark. This is a very tricky iterated forcing argument. To tell the truth, we asked this question as a curiosity, not really believing in the feasibility of an answer.

To be able to state the next problem we need the following

Definition 46 *For every graph G let* $\mathrm{Fin}(G)$ *be the set (of isomorphism classes) of finite subgraphs.*

Note that $|\,\mathrm{Fin}(G)| \leq 2^{\aleph_0}$.

Lemma 47 *There is a cardinal $\kappa(*)$ such that if* $\mathrm{Chr}(G) \geq \kappa(*)$ *then for every λ there is a graph G' with* $\mathrm{Chr}(G') \geq \lambda$ *and* $\mathrm{Fin}(G') \subseteq \mathrm{Fin}(G)$.

Proof: Call a set S of finite graphs bounded if there is a cardinal λ such that there is no graph G with $\mathrm{Chr}(G) \geq \lambda$ and $\mathrm{Fin}(G) \subseteq S$. For a bounded S let $\lambda(S)$ be the smallest bound. Then

$$\kappa(*) = sup\{\lambda(S)^+ : S\text{bounded}\}$$

satisfies the requirement.

Problem 48 (W. Taylor[34, 35]) *Is $\kappa(*) = \omega_1$?*

Note that $\kappa(*)$ is defined like the Hanff number of a logic. Erdős liked and popularized this problem, he stated it in many problem papers, e. g. [6, 7, 8, 13] though he usually stated his own problems only. The following two theorems contain significant results on this problem.

Theorem 49 (Komjáth [26]) *It is consistent that $\kappa(*) \geq \omega_2$.*

Proof in outline: Start with a model of ZFC+GCH+$\diamondsuit$. Let P be a forcing adding a Cohen real. Then as it is known there is an undominated $f : \omega \to \omega$ (not dominated by any $g : \omega \to \omega$ in the ground model). Then GCH and $\diamondsuit$ remain true after forcing with P. Prepare Shelah's forcing Q for this function f, call the graph obtained by X.

We claim that after forcing with $P * Q$ the following is true:

If $\mathrm{Fin}(Y) \subseteq \mathrm{Fin}(X)$ then $\mathrm{Chr}(Y) \leq \aleph_2$. (This clearly implies the claim.)

Let Y be such that $\mathrm{Fin}(Y) \subseteq \mathrm{Fin}(X)$. As $|P \times Q| = \aleph_1$, Y is the union of $\aleph_1$ graphs Z such that Z is in the ground model and

$$\mathrm{Fin}(Z) \subseteq \mathrm{Fin}(X)$$

Now for any of the above Z, $\mathrm{Chr}(Z) \leq \omega$ otherwise the function f_Z defined on the previous page would be in the ground model and would majorize f. Then

$$\mathrm{Chr}(Y) \leq \aleph_0^{\aleph_1} = \aleph_2.$$

The next theorem gives a bound from the other direction.

Theorem 50 (Komjáth,[26]) *It is consistent that* $\kappa(*) \leq \omega_2$.

Hint for the proof: Start with a model of ZFC+GCH, and collapse $\kappa(*)$ to ω. The claim is true in the resulting model.
Now we turn to an other problems treated in [15]

Definition 51

$$f^1_G(n) := \max\left\{\min\{|Z| : Z \subseteq V \wedge |Z| = n \wedge G[A - Z] \text{ is bipartite }\}\right\}.$$

Remark. $f^1_G(n) \leq f(n)$ means that we can omit $f(n)$ vertices from each n element set that G is bipartite.

We proved in [15] the following result.

Theorem 52 *For every* $\epsilon > 0$*, for every* κ *there is a graph* G *with* $\mathrm{Chr}(G) > \kappa$ *and* $f^1_G(n) \leq \epsilon n$.

Remark. This is basicly best possible. I omit the details of this.

Problem 53 *Can the theorem be proved under the additional condition that* $|G| = \kappa$ *?*

The next problem deals with an analogous question for edge omission instead of vertex omission.

Definition 54

$$f^2_G(n) := \max\left\{\min\{|E'| : G[A] - E' \text{ is bipartite }\} : A \subseteq V \wedge |A| = n\right\}.$$

Remark. $f^2_G(n) \leq f(n)$ means that we can omit $f(n)$ edges from each n element subgraph so that the remaining graph is bipartite. Here we know surprisingly little.

Theorem 55 ([15])

(i) There exsits a graph G *with* $\mathrm{Chr}(G) > \omega$ *such that* $f^2_G(n) \leq cn^{3/2}$.

(ii) If $\mathrm{Chr}(G) > \omega$ *then* $\exists k \in \omega$ *such that* $f^2_G(n) \geq \frac{1}{k}n$ *for infinitely many* $n \in \omega$.

We could not fill up the enormous gap between (i) and(ii). We expected a positive answer to the next problem:

Problem 56 ([15]) *Does there exist a graph* G *with* $\mathrm{Chr}(G) > \omega$, $f^2_G(n) < cn$ *for some real number* c *?*

However here the real problem is finitary.

Problem 57 ([15]) *Is it true that for every* $f : \omega \to \omega$ *tending increasingly to infinity there is a graph* G *with* $\mathrm{Chr}(G) = \omega$ *and* $f^2_G(n) = O(f(n))$ *?*

Theorem 58 (Rödl, [30]) *The answer is affirmative for* 3*-uniform hypegraphs.*

For graphs—to the best of my knowledge—the following is the strongest known result:

Theorem 59 (Lovász [27]; Rödl [30]) *For every $2 \leq r \leq \omega$, and for large enough $m \in \omega$ there is a graph G^r_m such that $\mathrm{Chr}(G^r_m) \geq r+2$ and for some $c > 0$,*

$$f^2_{G^r_m}(n) \leq cn^{1-\frac{1}{r}}$$

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