$\mu-$Equicontinuity and weak convergence in zero-dimensional spaces

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Zero-dimensional spaces

- Let $\mathcal{A}$ be a finite set. We endow $\mathcal{A}^{\mathbb{Z}^d}$ with the cantor metric, for $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $i \in \mathbb{Z}^d$, x_i denotes the ith coordinate of x, and σ_i the shift in the *ith* direction. We will use $X \subset \mathcal{A}^{\mathbb{Z}^d}$ to denote compact subsets. These spaces are zero-dimensional.

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- A zero-dimensional dynamical system (ZDS) $T : X \to X$ is a continuous transformation (Cantor endomorphisms are ZDS). $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a subshift if it is compact and $\sigma-$invariant, shifts of finite type (SFT) are a special kind of subshifts. Let X be a shift space. We define a cellular automaton (CA) as a σ-commuting ZDS $\phi : X \to X$.

Equicontinuity

- We denote the **balls** with $B_n(x) := \left\{ z \mid d(x,z) \leq \frac{1}{2^{n+1}} \right\}$, and the **orbit balls** as $O_m(x) := \left\{ y \mid d(T^i(x), T^i(y) \leq \frac{1}{2^{m+1}} \ \forall i \in \mathbb{N} \right\}.$

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- T is equicontinuous iff the family $\left\{ T^i \right\}_{i \in \mathbb{N}}$ is equicontinuous.

mu-Equicontinuity

- We will denote Borel probability measures of X as μ (which are not necessarily invariant).
  A point x is a $\mu-$**equicontinuity** point of T if for all $m \in \mathbb{N}$, one has
  $$\lim_{n\to\infty} \frac{\mu(B_n(x) \cap O_m(x))}{\mu(B_n(x))} = 1.$$
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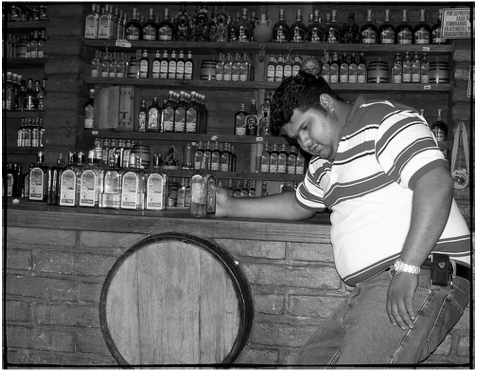
- There exists CA, with no equicontinuity points, that are $\mu-$equicontinuous for every $\sigma-$invariant measure.

Luis Perez

- My ex-neighbour



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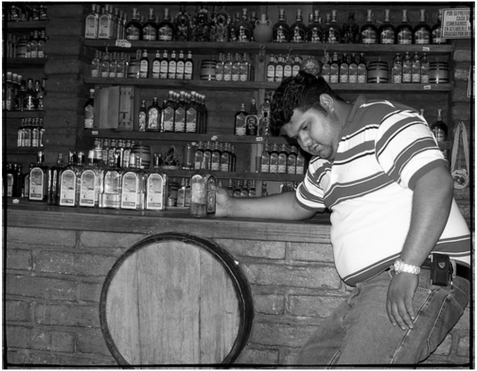

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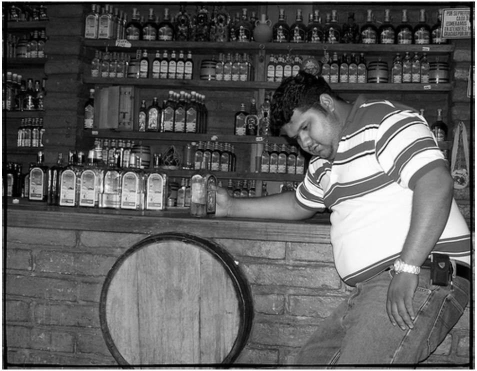

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- His son Luis Evo Perez was a bit different

Local periodicity

- We define the set of **locally periodic points,** $LP(T) := \left\{ x \mid (T^n x)_i \text{ is periodic } \forall i \in \mathbb{Z}^d \right\}$, and the set of **locally eventually periodic points,** $LEP(T) := \left\{ x \mid (T^n x)_i \text{ is eventually periodic } \forall i \in \mathbb{Z}^d \right\}$.

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- If X is a 1D SFT and μ a shift-invariant measure then $\mu-$equicontinuity $\Rightarrow \mu - LEP$. (G., generalization of a result by Gilman)
- If $\mu(\overline{\sigma - periodic\ points}) = 1$ and $\mu(\text{Eq. points of } \phi) = 1$, then ϕ is $\mu - LEP$. (G.)

Limiting behaviour of mu-LEP measures

- Blanchard-Tisseur (2000) studied the Cesaro limits of 1D shift-ergodic measures that gave equicontinuity points of a CA full-support. This maps are $\mu - LEP$ so the following is a generalization of one of their results.

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Theorem (G.)

Let $T : X \to X$ a $\mu - LEP$ ZDS. There exists a measure μ_∞, such that the Cesaro average $\frac{1}{n}\sum_{i=1}^{n} T^i(\mu) \to^w \mu_\infty$ (i.e. $\mu_n(B)$ converges for every ball B). Furthermore T is $\mu_\infty - LP$.

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- In their paper they asked the questions, when is μ_∞ $T-$ergodic, $\sigma-$ergodic or a measure of maximal entropy?

Shift-ergodic measures

Theorem (G.)

Let μ be a $\sigma-$ergodic measure and ϕ a $\mu - LEP\ CA$. The following are equivalent:

1) $\phi^n \mu \rightarrow^w \mu_\infty$

2) μ_∞ *is* $\sigma-$*ergodic*

3) $\phi^n \mu(O)$ *converges for every orbit ball* O.

Furthermore if ϕ is surjective then μ_∞ is $\sigma-$ergodic if and only if μ is invariant under ϕ.

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Furthermore if ϕ *is surjective then* μ_∞ *is* $\sigma-$*ergodic if and only if* μ *is invariant under* ϕ.

- Using this result we get that if X is a subshift with a unique measure of maximal entropy then μ_∞ is the measure of maximal entropy if and only if μ is the measure of maximal entropy and ϕ is surjective. Hence $\mu - LEP$ CA does no *randomize* measures.

Topological dynamical systems

- A topological dynamical system (TDS) is a continuous transformation on a compact metric space. The concepts of $\mu-$equicontinuity and $\mu - LEP$ can be defined for TDS.

Theorem (G.)

Let T be a TDS and μ a Borel probability measure. The following are equivalent:
1) T is $\mu-$equicontinuous
2) For every $\epsilon > 0$ there exists a compact set M (not necessarily invariant) such that $\mu(M) > 1 - \epsilon$ and $T\mid_M$ is equicontinuous.

Measure preserving transformations

- Using the previous characterization and a result by Huang-Lu-Ye (2009) we obtain the following result.

Theorem (Huang-Lu-Ye, and G.)

Let T be a $\mu-$equicontinuous $\mu-$ergodic TDS . We have that (T,μ) has discrete spectrum, so it is isomorphic (measure theoretically) to a rotation on a compact abelian group.

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- Not every $\mu-$equicontiunous measure preserving TDS has discrete spectrum. Nonetheless I am interested in seeing if you can characterize dynamical systems with discrete spectrum using equicontinuity-like properties.