A categorical invariant of flow equivalence of shifts

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The syntactic semigroup

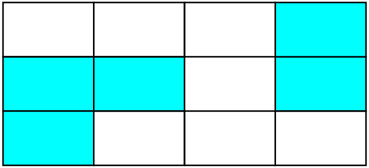
Let L be a language of A^+. The syntactic semigroup of L is the quotient of A^+ by the congruence

$$u \equiv_L v \Leftrightarrow C(u) = C(v)$$

where

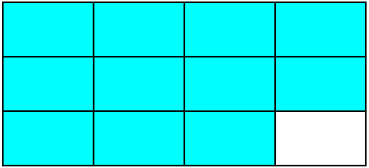
$$C(w) = \{(z,t) \in A^* \mid zwt \in L\}.$$

$A^+/\equiv_L$:



The syntactic semigroup

Let $\mathscr{X}$ be a subshift of $A^{\mathbb{Z}}$. If $\mathscr{X}$ is strictly contained in $A^{\mathbb{Z}}$, then the syntactic semigroup of $L(\mathscr{X})$, as a subset of A^+, is a semigroup with a zero.



The elements of $A^+ \setminus L$ form a class, which is the zero element

To avoid treating $A^{\mathbb{Z}}$ differently, we add a zero to the syntactic semigroup of $L(A^{\mathbb{Z}})$.

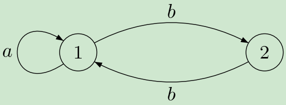
Notation for the syntactic semigroup (with zero): $S(\mathscr{X})$.

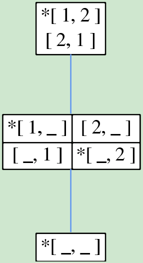
The syntactic semigroup

The transition semigroup of the Fischer cover of an irreducible sofic shift $\mathscr{X}$ is isomorphic to $S(\mathscr{X})$.

Example

Even shift





The syntactic semigroup

The transition semigroup of the Krieger cover of a sofic shift $\mathscr{X}$ is isomorphic to $S(\mathscr{X})$.

Example

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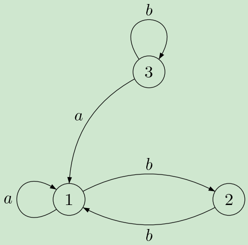

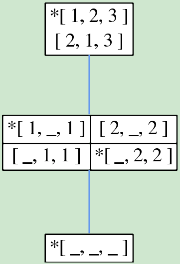

The Karoubi envelope

Let S be a semigroup. The Karoubi envelope of S is a category $K(S)$ defined by:

- the objects are the idempotents $e = e^2$ of S;
- the arrows $f \longrightarrow e$ are the triples (e, s, f) of elements of S such that $s = esf$;
- the composition of arrows is given by

$$(e, s, f)(f, t, g) = (e, st, g).$$

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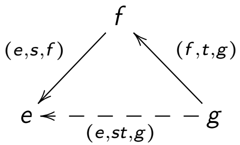

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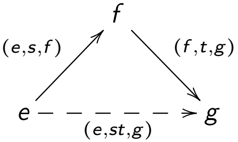

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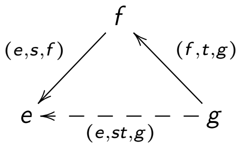

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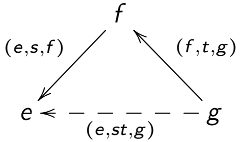
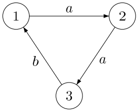
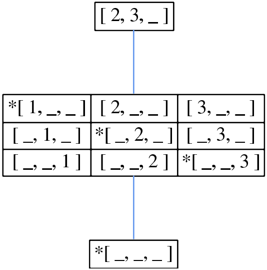


Note that the it suffices to work with $LU(S) = \{s \in S \mid s = esf, e = e^2, f = f^2\}$:

$$K(S) = K(LU(S))$$

Some remarks





- The Karoubi envelope of $S(\mathscr{X})$ is the Karoubi envelope of $S(\mathscr{X}) \setminus \{[2, 3, \_]\}$.
- The Karoubi envelope of an irreducible finite shift is equivalent to the Karoubi envelope of $\{0, 1\}$.
- The Karoubi envelope of the minimal shift is the trivial monoid.

Invariance of the Karoubi envelope

Denote by $\mathbb{K}(\mathscr{X})$ the Karoubi envelope of $S(\mathscr{X})$.

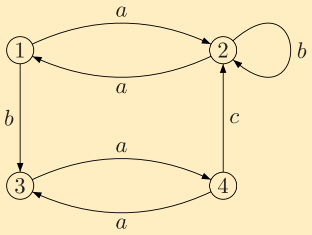
Theorem (Steinberg & AC)

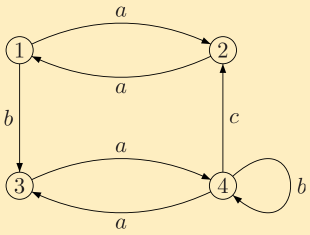
If $\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent, then $\mathbb{K}(\mathscr{X})$ and $\mathbb{K}(\mathscr{Y})$ are equivalent.

A "monoidal" example

Corollary

If $\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent shifts such that $S(\mathscr{X})$ and $S(\mathscr{Y})$ are monoids, then $S(\mathscr{X})$ and $S(\mathscr{Y})$ are isomorphic.





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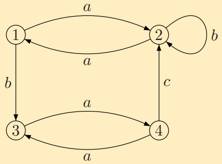
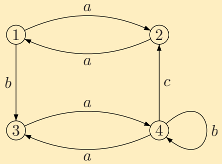
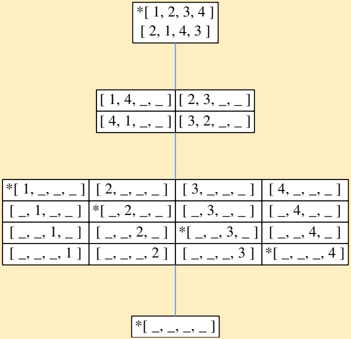
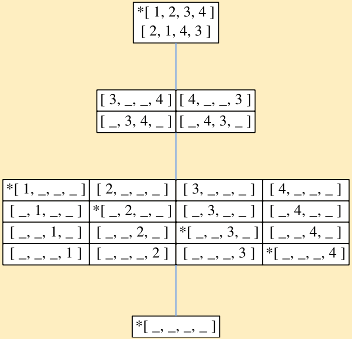




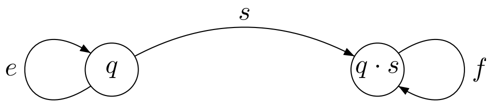




Action of $\mathbb{K}(\mathscr{X})$

- Extend the Krieger cover to a complete automaton by adding a sink state $\emptyset$.
- Let Q be the set of vertices of the extended automaton.
- An arrow (e, s, f) of $\mathbb{K}(\mathscr{X})$ defines the mapping

$$A_{\mathscr{X}}(e,s,f)\colon \begin{array}{ccc} Q\cdot e & \to & Q\cdot f \\ q & \mapsto & q\cdot s \end{array}$$



The correspondence $(e, s, f) \mapsto A_{\mathscr{X}}(e, s, f)$ is a functor.

Invariance of the action

Theorem

If $\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent shifts, then the actions $\mathbb{A}_{\mathscr{X}}$ and $\mathbb{A}_{\mathscr{Y}}$ are equivalent.

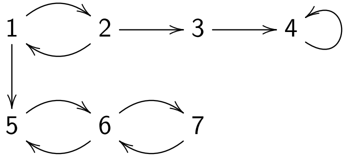
That is, there is an equivalence functor $F\colon \mathbb{K}(\mathscr{X}) \to \mathbb{K}(\mathscr{Y})$ and a natural isomorphism $\eta\colon \mathbb{A}_{\mathscr{X}} \to \mathbb{A}_{\mathscr{Y}} \circ F$ such that...

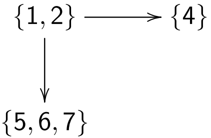
$$\begin{array}{ccc} Q(\mathscr{X})e & \xrightarrow{\eta_e} & Q(\mathscr{Y})F(e) \\ {\scriptstyle \mathbb{A}_{\mathscr{X}}(e,s,f)}\downarrow & & \downarrow{\scriptstyle \mathbb{A}_{\mathscr{Y}}(F(e,s,f))} \\ Q(\mathscr{X})f & \xrightarrow{\eta_f} & Q(\mathscr{Y})F(f). \end{array}$$

The proper comunication graph

The proper communication graph of a (directed) graph G is defined as follows:

1. take the set $PC(G)$ of nontrivial (i.e. having at least one edge) strongly connected components of G,
2. for $C_1, C_2 \in PC(G)$, make $C_1 \leq C_2$ if there is a path from a vertex of C_1 to a vertex of C_2,
3. the proper communication graph of G is the acyclic graph induced by the poset $(PC(G), \leq)$.

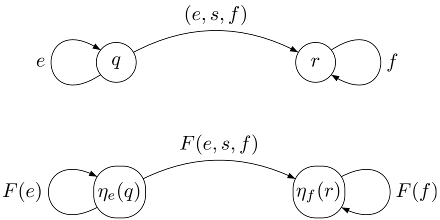




Theorem (Bates, Eilers & Pask, 2011)

The proper communication graph of the Krieger cover of a sofic shift is a flow equivalence invariant.

The proper comunication graph



Let $LU(\mathscr{X}) = \{s \in S(\mathscr{X}) \mid s = esf,\ \text{for some idempotents } e, f \in S(\mathscr{X})\}$.

Let q, r be vertices stabilized by some idempotent.

Then $q \leq r$ if and only if $rLU(\mathscr{X}) \subseteq qLU(\mathscr{X})$.

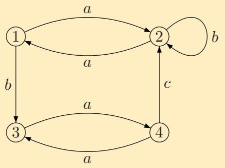
Generalization

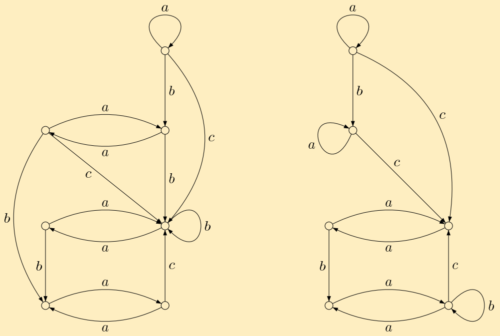
The poset

$$\{qLU(\mathscr{X}) \mid q \text{ is stabilized by some idempotent}\}$$

is invariant under flow equivalence.

The action: an example





The action on the Fischer cover

Theorem

If $\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent synchronizing shifts, then $\mathbb{K}(\mathscr{X})$ and $\mathbb{K}(\mathscr{Y})$ have equivalent actions over the corresponding Fischer covers.

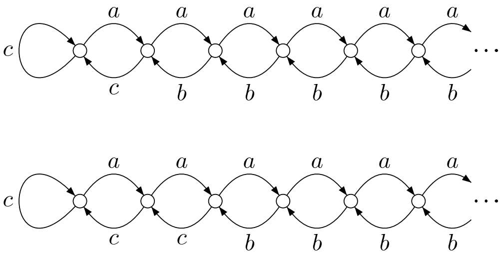
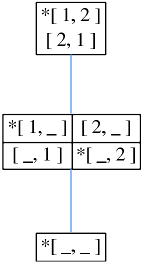
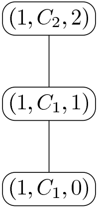


Figure: Fischer covers of two synchronizing shifts.

In the first shift, the rank of every block of the shift is one or infinite.
In the second shift, the word *ac* acts as an idempotent of rank two.

An invariant hailing from Green's relations



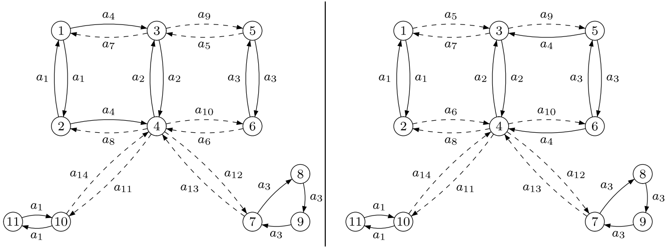


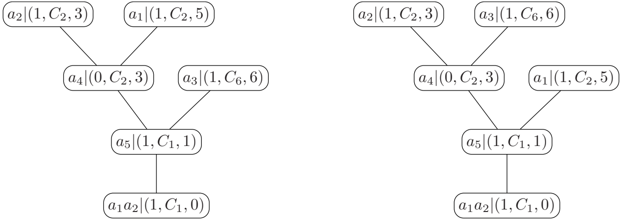
Legend for the label (i, G, r) of a $\mathcal{D}$-class D:

- $i = 1$ if D is regular, $i = 0$ otherwise;
- G is the (isomorphism class) of the Schützenberger group of D;
- r is the rank of the action of the elements of the $\mathcal{D}$-class on the Fischer cover.

An analogous result holds replacing the Fischer cover by the Krieger cover, which is valid for all shifts (not necessarily sofic).

An invariant hailing from Green's relations





The poset of subsynchronizing subshifts of a sofic shift

N. Jonoska constructed a conjugacy invariant for a reducible shift $\mathscr{X}$, the poset of subsynchronizing subshifts of $\mathscr{X}$.

These subshifts are the unions of subshifts whose finite blocks are factors of right contexts of a magic word for $L(\mathscr{X})$
(a word m is magic if m is synchronizing and $mA^*m \cap L(\mathscr{X}) \neq \varnothing$).

Using our main result, we showed that this poset is a flow invariant.

Classification of Markov-Dyck shifts

Let G be a finite (directed graph). Let G' be the graph obtained from G by adjoining to each edge $x : u \to v$ an inverse edge $x^{-1} : v \to u$, establishing a bijection $x \in E(G) \mapsto x^{-1} \in E(G)^{-1}$.

Presentation of the graph inverse semigroup P_G

1. P_G is generated by $E(G) \cup E(G)^{-1} \cup \{0\} \cup \{1_v \mid v \in V(G)\}$.
2. the generators are subject to the relations
   (a) 1_v are local identities
   (b) $xx^{-1} = 1_{\alpha x}, \quad x \in E(G)$
   (c) $xy^{-1} = 0$ if $y \neq x, \quad x, y \in E(G)$

- P_G is an inverse semigroup.
- The words, over the alphabet $E(G) \cup E(G)^{-1}$, whose image in P_G is not 0, define a shift, the Markov-Dyck shift D_G.

Classification of Markov-Dyck shifts

If the out-degree of a vertex is always at least one, then the semigroup with zero P_G is generated by $E(G) \cup E(G)^{-1}$.

Lemma

Suppose each vertex of G has out-degree at least one.
Then P_G is the syntactic semigroup of D_G if and only if G has no vertex of in-degree exactly one.

Theorem

Let G and H with out-degree always at least one and in-degree never one.

$$D_G \text{ is flow equivalent to } D_H \Leftrightarrow G \cong H.$$

Proof.

- D_G flow equivalent to $D_H \Rightarrow \mathbb{K}(P_G)$ equivalent to $\mathbb{K}(P_H)$
- $\mathbb{K}(P_G)$ equivalent to $\mathbb{K}(P_H) \Rightarrow G \cong H$ □