A maximal entropy stochastic process for a timed automaton

Automata Theory and Symbolic Dynamics Workshop
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Problem statements

Theoretical problem statement

Lift the Shannon/Parry Markov chain of a strongly connected finite graph to the timed automata settings.
(aka MME of an irreducible SFT)

Practical problem statement

Generate **quickly** and as **uniformly** as possible runs of a timed automaton.

- quickly: Step by step simulation as with a finite state Markov Chain $\rightarrow$ Stochastic Process Over Runs (SPOR)
- $\approx$ uniformly $\rightarrow$ SPOR of maximal entropy + asymptotic equipartition property.

Motivations

Possible applications of (quasi) uniform random simulation

- Proportional model checking e.g. more than 65 per cent of the runs satisfies a formula with probability of error ≤ 0.01.
- Fast (quasi) uniform generation in certain classes of permutation e.g. alternating permutations.

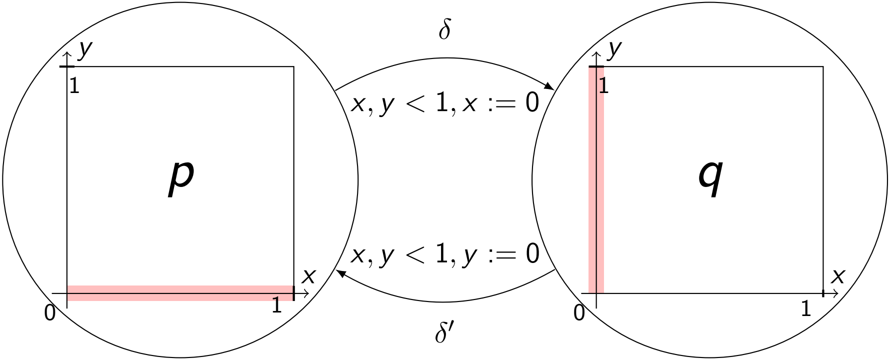
Other possible applications

- Compression of timed words in a timed regular language.

Outline

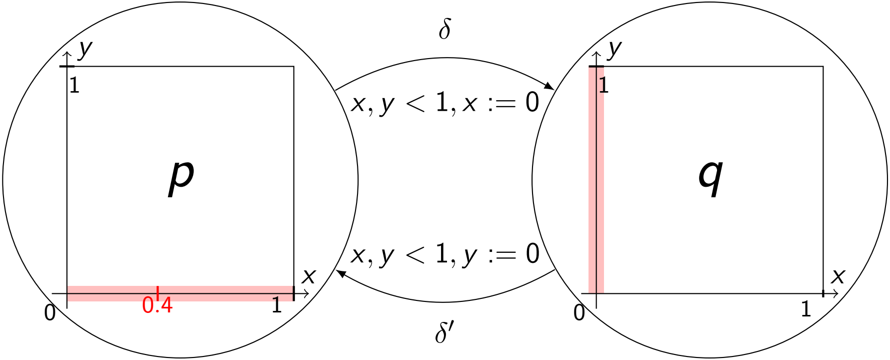
Timed region graph

- Timed region graph (TRG)= Timed automaton without labels on transitions, initial and final set of states = entry regions.



A run of the timed region graph

$$(p,(0.4,0)) \xrightarrow{0.3,\delta} (q,(0,0.3)) \xrightarrow{0.2,\delta'} (p,(0.2,0)) \xrightarrow{0.6,\delta} (q,(0.6,0))$$



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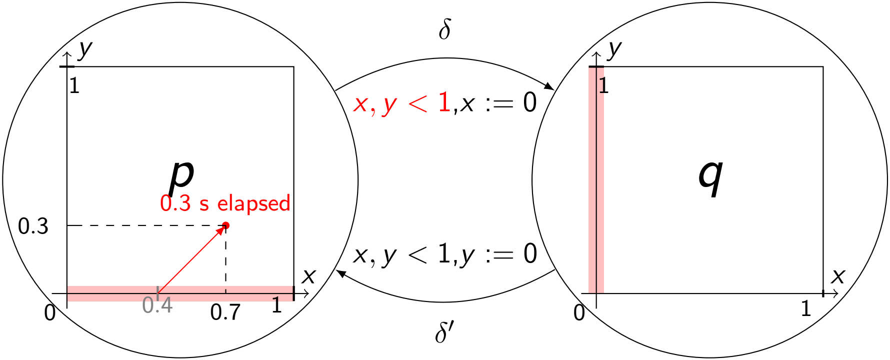
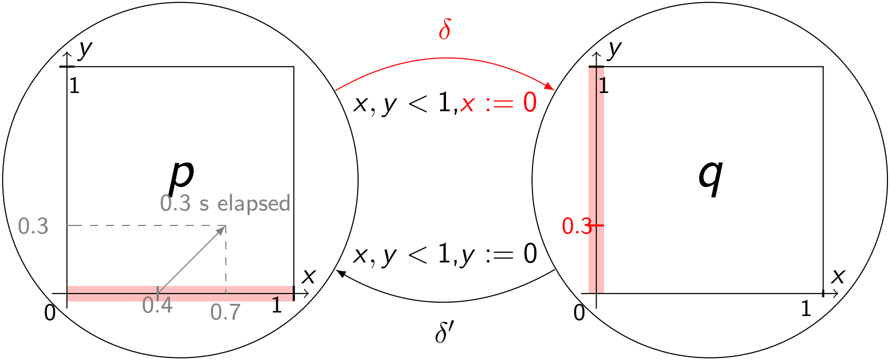


$x = 0.7 < 1$ and $y = 0.3 < 1$, the guard is satisfied.

A run of the timed region graph

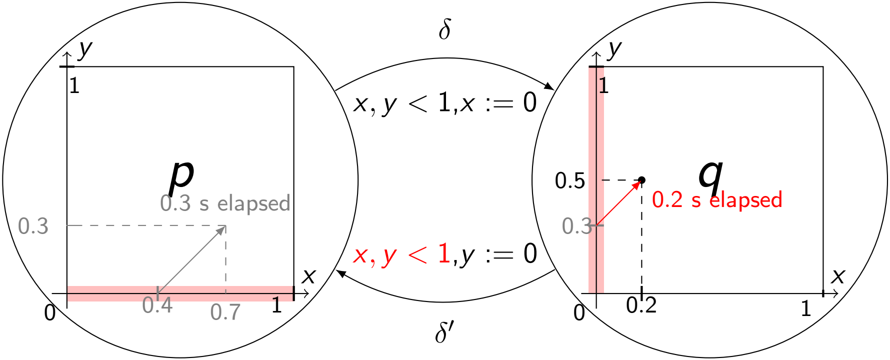
$(p,(0.4,0)) \xrightarrow{0.3,\delta} (q,(0,0.3)) \xrightarrow{0.2,\delta'} (p,(0.2,0)) \xrightarrow{0.6,\delta} (q,(0.6,0))$



x is reset while the transition is fire, $y = 0.3$ is unchanged.

A run of the timed region graph

$$(p,(0.4,0)) \xrightarrow{0.3,\delta} (q,(0,0.3)) \xrightarrow{0.2,\delta'} (p,(0.2,0)) \xrightarrow{0.6,\delta} (q,(0.6,0))$$



$x = 0.2 < 1$ and $y = 0.5 < 1$, the guard is satisfied.

A run of the timed region graph

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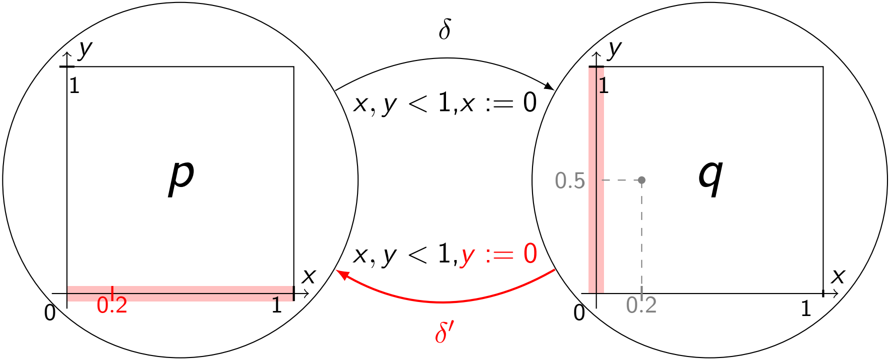

y is reset while the transition is fired, x is unchanged.

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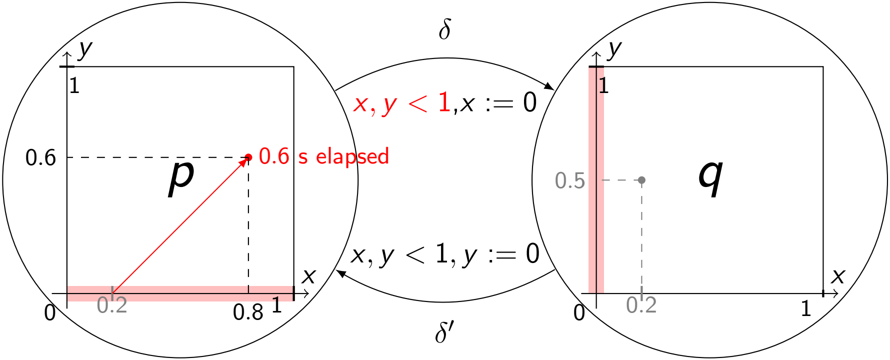

$x = 0.8 < 1$ and $y = 0.6 < 1$, the guard is satisfied.

A run of the timed region graph

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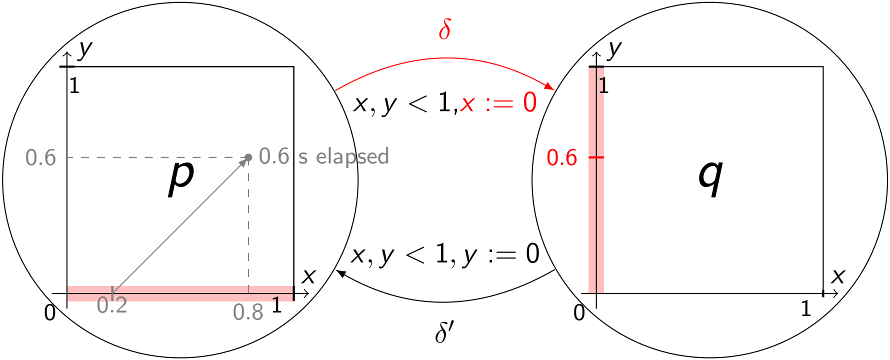


x is reset while the transition is fire, $y = 0.6$ is unchanged.

Measuring runs

An infinite transition system

- Dense set of states $(q, \vec{x}) \in \mathbb{S}$.
- Dense set of timed transitions $(t, \delta) \in \mathbb{A}$.
- Successor action of $\mathbb{A}$ on $\mathbb{S}$: $s' = s \triangleright \alpha$.
- Runs $s_0 \xrightarrow{\alpha_0} s_1 \cdots \xrightarrow{\alpha_{n-1}} s_n$ denoted by $[s_0, \alpha_0, \cdots, \alpha_{n-1}]$

Integrating over states, timed transition and runs

- Integrating over $\mathbb{A}$: $\int_{\mathbb{A}} f(\alpha) d\alpha = \sum_{\delta \in \Delta} \int_0^M f(t, \delta) dt$.
- Integrating over $\mathbb{S}$: $\int_{\mathbb{S}} f(s) ds = \sum_{q \in Q} \int_{\mathbf{r}_q} f(q, \mathbf{x}) d\mathbf{x}$.
- Integration over runs: $\int_{\mathbb{S} \times \mathbb{A}^n} f([s_0, \alpha_0, \cdots, \alpha_{n-1}]) ds_0 d\alpha_0 \cdots d\alpha_{n-1}$ where $f(\perp) = 0$
- $\mathtt{Vol}(\mathtt{Runs}_n) = \int_{\mathbb{S} \times \mathbb{A}^n} 1_{[s_0, \alpha_0, \cdots, \alpha_{n-1}] \neq \perp} ds_0 d\alpha_0 \cdots d\alpha_{n-1}$

Stochastic Process Over Runs (SPOR)

A SPOR (Semi Markov)

- Initial density on states: $p_0 : \mathbb{S} \to \mathbb{R}^+$ such that $\int_{\mathbb{S}} p_0(s)ds = 1$.
- Conditional density on timed transition $\mathbb{A}$: $\int_{\mathbb{A}} p(\alpha|s)d\alpha = 1$.

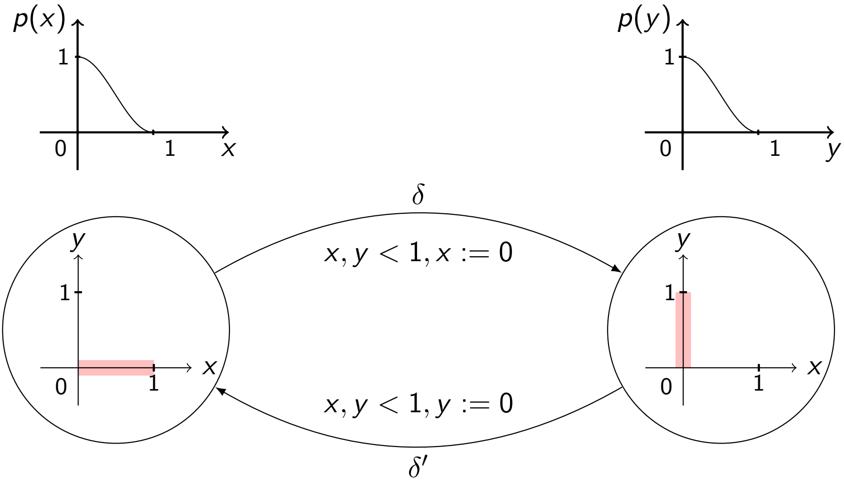
Induced probability density function (PDF) on $\texttt{Runs}_n$

- Chain rules:
  $p_n([s_0, \alpha_0, \cdots, \alpha_{n-1}]) = p_0(s_0)p(\alpha_0|s_0)\cdots p(\alpha_{n-1}|s_{n-1})$
- Probability of a set of runs $R \subseteq \texttt{Runs}_n$:

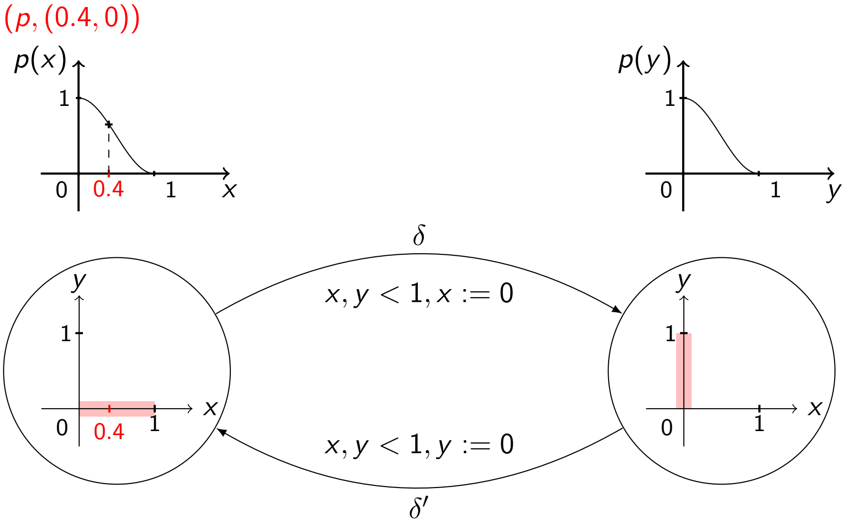
$$P(R) = \int_R p_n(r)dr$$

- $P(\texttt{Runs}_n) = 1$.

An initial PDF on state

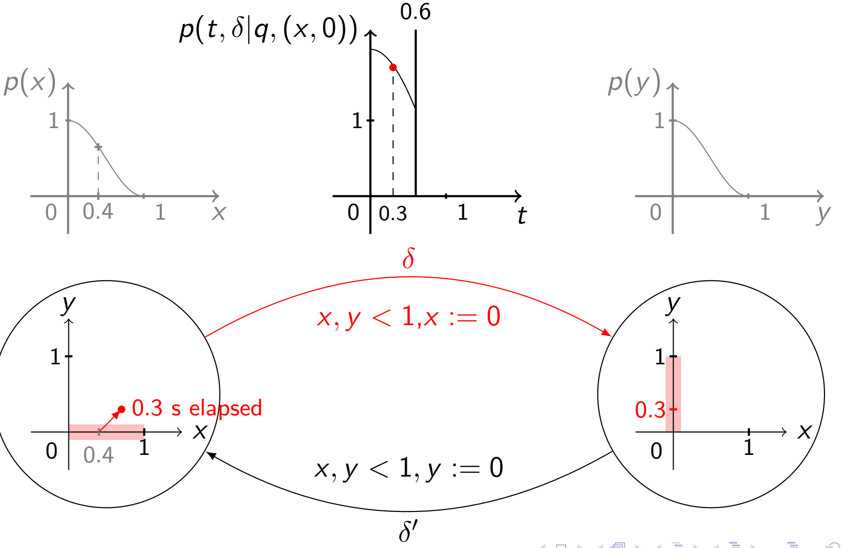


Choosing a starting state according to the PDF.



Choosing a timed transition (transition and delay).

$$(p,(0.4,0)) \xrightarrow{0.3,\delta} (q,(0,0.3))$$



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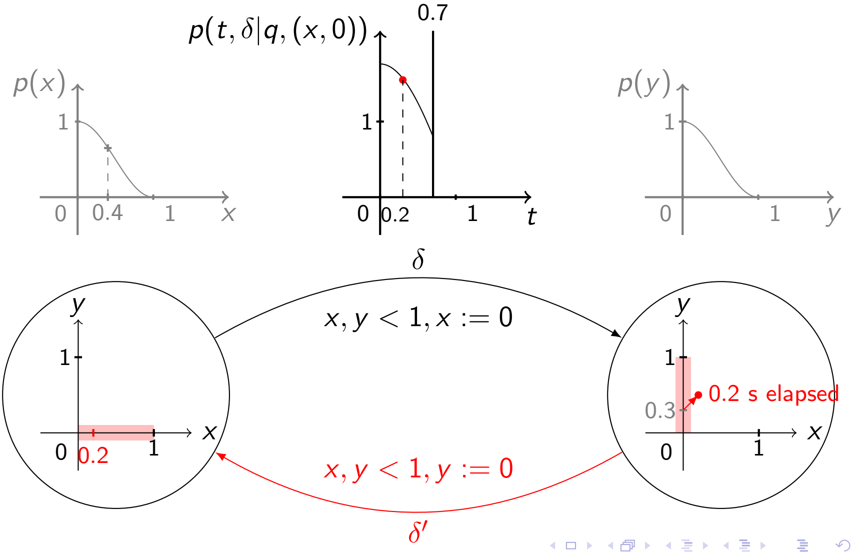

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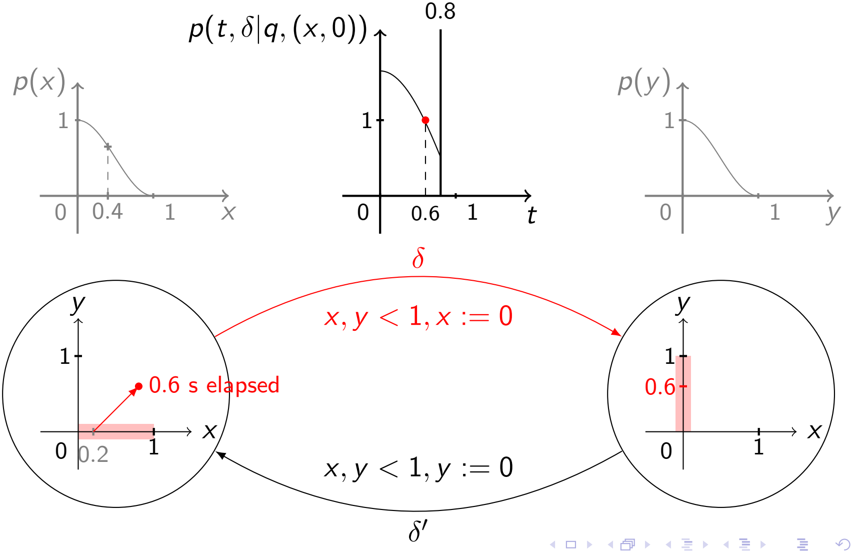

Problem statement, a recap

Problem statement (Unformal)

Describe a SPOR that generates as uniformly as possible runs in a timed region graph?

$$p_n(r) \approx \frac{1}{\mathtt{Vol}(\mathtt{Runs}_n)} \quad \text{For "almost" every run } r.$$

Solution based on entropy

Max entropy = as uniformly as possible.

Outline

Entropy

- Entropy of runs :

$$\mathcal{H} = \lim_{n \to +\infty} \frac{1}{n} \log_2(\texttt{Vol}(\texttt{Runs}_n))$$

- Entropy of a SPOR Y:

$$h(Y) = \lim_{n \to +\infty} -\frac{1}{n} \int_{\texttt{Runs}_n} p_n(r) \log_2 p_n(r) dr$$

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Theorem 1

There exists Y^* of maximal entropy $h(Y^*) = \mathcal{H}$ (described later).

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Asymptotic equipartition property

Y^* satisfies $-\frac{1}{n} \log_2 p_n(r) \to h(Y^*)$ almost surely.

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Theorem 1

There exists Y^* of maximal entropy $h(Y^*) = \mathcal{H}$ (described later).

Asymptotic equipartition property

Y^* satisfies $-\frac{1}{n} \log_2 p_n(r) \to h(Y^*)$ almost surely.

Solution of the problem

Most of the runs have a quasi uniform probability to occur:
$p_n(r) \approx 2^{-nh(Y*)} = 2^{-n\mathcal{H}} \approx 1/\texttt{Vol}(\texttt{Runs}_n)$.

The operator Ψ of Asarin and Degorre (FORMATS 2009)

The operator Ψ (new notation)

For $f : \mathbb{S} \to \mathbb{R}$, $s \in \mathbb{S}$:

$$\Psi f(s) = \int_{\alpha \in \mathbb{A}} f(s \triangleright \alpha) d\alpha \quad \text{with } f(\bot) = 0$$

New functional space for Ψ: $L^2(\mathbb{S})$

Square summable functions: $f \in L^2(\mathbb{S})$ if $\int_{\mathbb{S}} f^2(s) ds < +\infty$.
Scalar product: $\langle f, g \rangle = \int_{\mathbb{S}} f(s) g(s) ds$

Spectral radius, and corresponding eigenvectors

Theorem (Adapted from (Asarin, Degorre, FORMATS 2009) .)
$\mathcal{H} = \log_2(\rho)$.

Theorem (Perron-Frobenius like theorem)

1. *There exists a unique v positive a.e. such that $\Psi v = \rho v$ (unicity up to a scalar constant).*
2. *There exists a unique w positive a.e. such that $\Psi^* w = \rho w$, (unicity up to a scalar constant).*

Normalizing condition: $\langle w, v \rangle = \int_{\mathbb{S}} w(s)v(s)ds = 1$.

The maximal entropy SPOR

Main Theorem

The following PDFs defines an ergodic SPOR Y^* with maximal entropy $h(Y^*) = \mathcal{H}$:

$$p_0^*(s) = w(s)v(s) \quad (\Psi v = \rho v,\ \Psi^* w = \rho w,\ \int_{\mathbb{S}} w(s)v(s)ds = 1)$$

$$p^*(\alpha|s) = \frac{v(s \triangleright \alpha)}{\rho v(s)}$$

Analogy between timed and untimed case

untimed case	**timed case**
Graph G	Timed region graph $\mathcal{G}$
Paths	Runs
Markov chain on G	SPOR on $\mathcal{G}$
Adjacency matrix M	Operator Ψ on $L^2(\mathbb{S})$
Transposed matrix M^T	Adjoint operator Ψ^*
Spectral radius $\rho(M)$	Spectral radius $\rho(\Psi)$
$h(G) = \log_2(\rho(M))$	$\mathcal{H}(\mathcal{G}) = \log_2(\rho(\Psi))$
$Mv = \rho v$	$\Psi v = \rho v$
$wM = \rho w$ $(\Leftrightarrow M^T w^T = \rho w^T)$	$\Psi^* w = \rho w$
$\langle v, w \rangle = \sum v_i w_i = 1$	$\langle v, w \rangle = \int_S w(s)v(s)ds = 1$
$p_0^*(i) = v_i w_i$	$p_0^*(s) = w(s)v(s)$
$p^*(i \xrightarrow{\delta} j) = \frac{v_j}{\rho v_i}$	$p^*(\alpha \vert s) = \frac{v(s \triangleright \alpha)}{\rho v(s)}$

Hypotheses and proof details

The D-Weak progress condition (D-WPC)

On each path of length $\geq D$ all the clocks are reset at least once.

Lemme: kernel for Ψ^n, (HSIO)

If the D-WPC is satisfied then for $n \geq D$, there exists $k_n \in L^2(\mathbb{S} \times \mathbb{S})$ s.t.

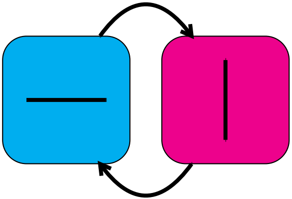
$$\Psi^n(f)(s) = \int_{s' \in S} k_n(s, s')f(s')ds', \; (\Psi^*)^n(f)(s') = \int_{s \in S} k_n(s, s')f(s)ds.$$

Thickness/forgetfulness and irreducibility of Ψ.

For strongly connected timed graph satisfying the D-WPC.

- $\mathcal{H} > -\infty$
- for all $s, s' \in \mathbb{S}$, there exists n such that $s \to^n s'$.
- for $q, q' \in Q$, there exists $n \in \mathbb{N}$ such that $k_n((q, \mathbf{x}), (q', \mathbf{x}'))$ is positive almost everywhere ($\Rightarrow$ irreducibility of Ψ).

Example



$$(x,0) \xrightarrow{t,\delta} (0,t) \text{ if } x+t<1$$
$$(0,y) \xrightarrow{t,\delta'} (t,0) \text{ if } y+t<1$$

- Eigenvector equations: $\Psi_a(v_1) = \rho v_1 \quad \Psi_b^*(w_1) = \rho w_1$; $\Psi_b(v_2) = \rho v_2 \quad \Psi_a^*(w_2) = \rho w_2$
- Same kernel operator for both transitions : $\Psi(v_i)(x) = \int k(x,t) v_i(t) dt$ where $k(x,t) = \mathbf{1}_{0 \le x+t<1}$.
- Ψ^* has kernel $k^*(x,t) =_{def} k(t,x) = k(x,t)$.
- Solutions $\rho = \frac{2}{\pi}$, $v_1 : x \mapsto \cos(\frac{\pi x}{2})$ $(= v_2 = Cw_1 = Cw_2)$.
- $p_0^*[B,(x,0)] = \cos^2(\frac{\pi x}{2}) \quad (= p_0^*[R,(0,y)])$.
  $p^*[t,a|B,(x,0)] = \frac{\pi}{2} \frac{\cos(\frac{\pi}{2}t)}{\cos(\frac{\pi}{2}x)} \mathbf{1}_{t \le 1-x} \quad (= p^*[t,b|R,(0,y)])$

Outline

Stochastic process over runs

The maximal entropy SPOR

Conclusion

What we have seen

- SPOR for timed region graph.
- Entropy for SPOR and TRG.
- Operator Ψ of Asarin and Degorre adapted to $L^2(\mathbb{S})$.
- Maximal entropy SPOR defined with ρ, v and w.
- Asymptotic equipartition property.

What we have not spoken about

- Stochastic operator φ for Y^*: similar to the transition probability matrix of a finite state Markov Chain.
- Stationarity and ergodicity of Y^*.
- Generation of timed word with a SPOR.
- Symbolic dynamics interpretation/vocabulary (bi-infinite runs, maximal entropy shift invariant measure...).

Future work

To do:

- Compute ρ, v, w numerically (Iterative methods) and symbolically (Solve integral equations).
- Remove the D-WPC.
- Describe the steady state analysis for other SPOR than Y^*.
- Correct the non uniformity of Y^*.

Possible applications:

- Proportional model checking.
- Fast (quasi) uniform generation of permutations.
- Compression and coding.