# Characterizing some limit sets of Cellular Automata

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June 6th 2013

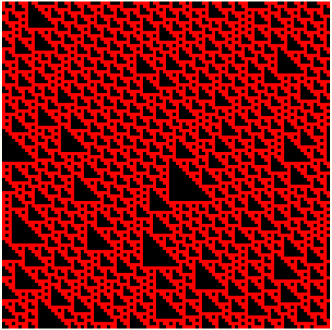
# Cellular Automata

## Dynamical System

Space: $\Sigma^{\mathbb{Z}}$ (fullshift), $\Sigma$ finite
Block map: $F : \Sigma^{[-r;r]} \to \Sigma$

$$\begin{array}{rccc} f: & \Sigma^{\mathbb{Z}} & \to & \Sigma^{\mathbb{Z}} \\ & f(x)_i & = & F(x_{i-r}, \ldots, x_{i+r}) \end{array}$$

$f$ 1-block: $r = 0$



## Symbolic dynamics

Shift: $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$, $\sigma(x)_i = x_{i-1}$
Hedlund: $f$ block map iff $f$ continuous and $f \circ \sigma = \sigma \circ f$

# Limit set and stability

## Definition

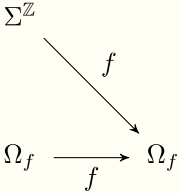
$$\Omega_f = \bigcap_{n \in \mathbb{N}} f^n(\Sigma^{\mathbb{Z}})$$

$\Omega_f$ is compact ($f$ continuous) and $\sigma$-invariant ($f$ $\sigma$-commuting): A subshift
*"configurations reachable arbitrary late in its evolution"*

## Stability (Maass'95)

$f$ stable iff $\exists N, \Omega_f = f^N(\Sigma^{\mathbb{Z}})$
$f$ unstable otherwise

# $f$ stable: $f \leftarrow f^N$




## Consequences

$f$ stable $\Rightarrow$ $\Omega_f$ is a factor of a fullshift.
Boyle'84: $\mathbf{X}$ factor of a fullshift $\Leftrightarrow$ $\mathbf{X}$ sofic, mixing and with a receptive fixed point.

# The problem

**Open problem (Maass'95, Boyle OPSD)**

Get a characterization of stable limit sets of CA.
*Can these be all fullshift factors ?*

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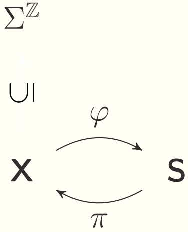
**Conjecture (BGK'11)**

$\mathbf{X}$ limit set of a stable CA iff:

- $\mathbf{X}$ factor of a fullshift
- $\exists\varphi : \mathbf{X} \to \mathbf{S}, \exists\pi : \mathbf{S} \to \mathbf{X}$, $\mathbf{S}$ SFT
  ($\mathbf{X}$ is weakly-conjugate to an SFT)

$\Leftarrow$ Already done: Boyle'84
$\Rightarrow$ Open question

# Proof of $\Leftarrow$




- $\exists \varphi : \mathbf{X} \to \mathbf{S}, \exists \pi : \mathbf{S} \to \mathbf{X}$
- **S** SFT factor of a fullshift

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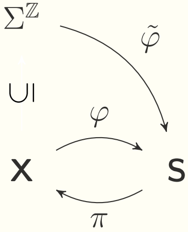

- $\exists\varphi : \mathbf{X} \to \mathbf{S}, \exists\pi : \mathbf{S} \to \mathbf{X}$
- $\mathbf{S}$ SFT factor of a fullshift
- Boyle extension lemma: $\exists\tilde{\varphi} : \Sigma^{\mathbb{Z}} \to \mathbf{S}$, $\tilde{\varphi}_{|\mathbf{X}} = \varphi$

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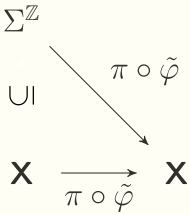


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- Boyle extension lemma: $\exists \tilde{\varphi} : \Sigma^{\mathbb{Z}} \to \mathbf{S}$, $\tilde{\varphi}_{|\mathbf{X}} = \varphi$
- $f = \pi \circ \tilde{\varphi}$

# What we will prove today

## Theorem

**X** *limit set of a right-continuing almost-everywhere stable CA iff:*

- **X** *factor of a fullshift*
- $\exists \varphi : \mathbf{X} \to \Sigma_R, \exists \pi : \Sigma_R \to \mathbf{X}$, $\Sigma_R$ *minimal right-resolving cover of* **X** *(right Fischer cover)*

## Facts

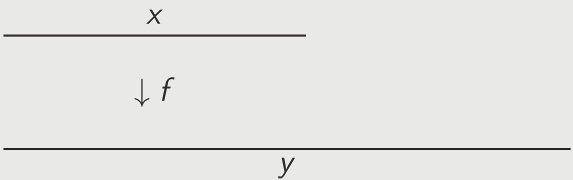
All known examples fall under this characterization.

# Right-resolving

## Definition

$f : \mathbf{X} \to \mathbf{Y}$ 1-block factor map. $f$ is right-resolving if $\forall x \in \mathbf{X}, y \in \mathbf{Y}, f(x)_{(-\infty;0]} = y_{(-\infty;0]}$, there exists at most one $x'_1 \in \mathcal{A}(\mathbf{X})$ s.t. $x_{(-\infty;0]}x'_1x'_{[2;\infty)} \in \mathbf{X}$ and $F(x'_1) = y_1$.

## Picture



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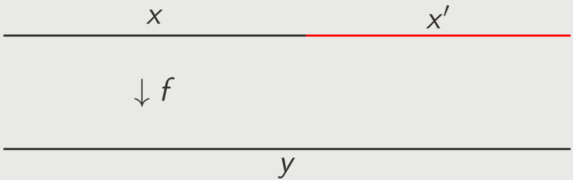

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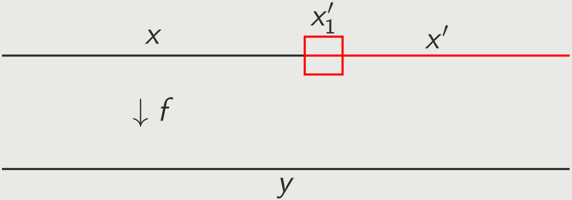

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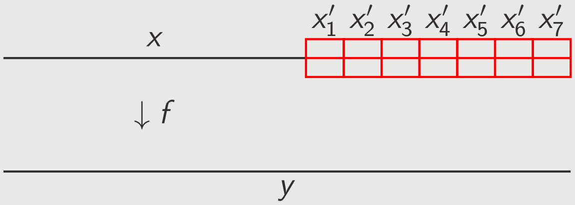

# Right-e-resolving

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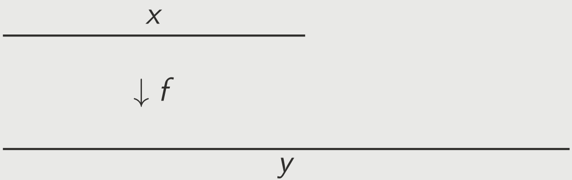

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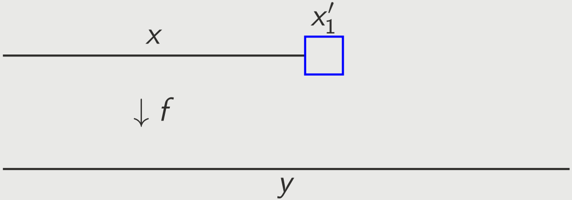

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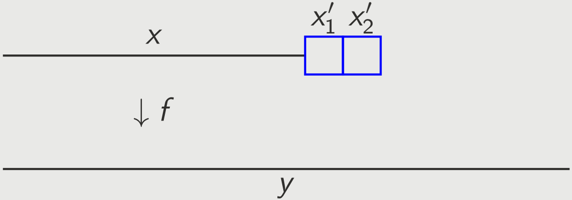

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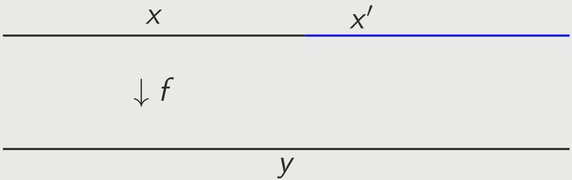

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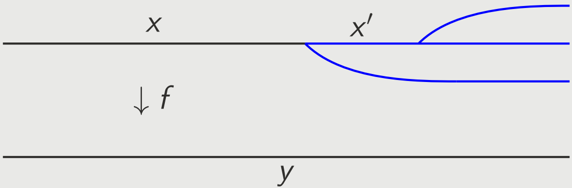

# Almost-everywhere

$x$ left-transitive if $\{\sigma^n(x), n \in \mathbb{N}\}$ dense in $\mathbf{X}$

### Definition

$f : \mathbf{X} \to \mathbf{Y}$ 1-block factor map. $f$ is right-e-resolving a.e. if $\forall x \in \mathbf{X}, y \in \mathbf{Y}$, $x$ left-transitive in $\mathbf{X}$, $f(x)_{(-\infty;0]} = y_{(-\infty;0]}$, there exists at least one $x'_1 \in \mathcal{A}(\mathbf{X})$ s.t. $x_{(-\infty;0]}x'_1x'_{[2;\infty)} \in \mathbf{X}$ and $F(x'_1) = y_1$.

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**Definition**

$f$ is right-continuing (a.e.) if $f$ is conjugate to a right-e-resolving (a.e.) factor map.

**Properties**

$f : \mathbf{X} \to \mathbf{Y}$, $\mathbf{X}$ SFT.

- right-resolving $\Rightarrow f$ right-e-resolving a.e.
- (right-resolving $\Rightarrow f$ right-e-resolving) $\Leftrightarrow \mathbf{Y}$ SFT

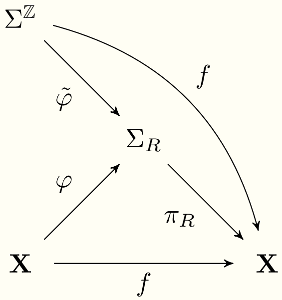
# The converse

**Theorem (Boyle-Tuncel'84)**

*If* $\varphi : \mathbf{X} \to \Sigma_R$, $\Sigma_R$ *SFT*, $\Sigma$ *SFT s.t.* $\mathbf{X} \subseteq \Sigma$ *then* $\exists \tilde{\varphi} : \Sigma \to \Sigma_R$, $\tilde{\varphi}_{|\mathbf{X}} = \varphi$, $\tilde{\varphi}$ *right-continuing.*

**Corollary**

$f = \pi_R \circ \tilde{\varphi}$ is right-continuing a.e.



## Definition

$f$ *right-e-resolving almost-everywhere stable CA* if $f : \Sigma^{\mathbb{Z}} \to \Omega_f$ is a *right-e-resolving almost-everywhere factor map*.

## Theorem

**X** *limit set of a right-e-resolving almost-everywhere stable CA then:*

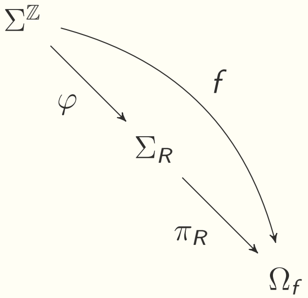
- **X** *factor of a fullshift*
- $\exists \varphi : \mathbf{X} \to \Sigma_R, \exists \pi : \Sigma_R \to \mathbf{X}$, $\Sigma_R$ *minimal right-resolving cover of* **X**

## Remark

We just proved the converse!

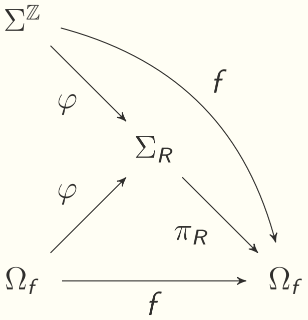
## Theorem

*If $f$ right-e-resolving a.e. then $\exists\varphi : \Sigma^{\mathbb{Z}} \to \Sigma_R$ s.t. $f = \pi_R \circ \varphi$.*



## Theorem

*If f right-e-resolving a.e. then* $\exists\varphi : \Sigma^{\mathbb{Z}} \to \Sigma_R$ *s.t.* $f = \pi_R \circ \varphi$.




## Lemma

$$\left.\begin{array}{l} \varphi(\Omega_f) \subseteq \Sigma_R \\ h(\Omega_f) = h(\Sigma_R) \\ \Omega_f, \Sigma_R \textit{ irreducible} \end{array}\right\} \Rightarrow \varphi(\Omega_f) = \Sigma_R$$

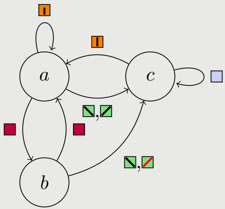
## Followers

$f : \mathbf{X} \to \mathbf{Y}$ a 1-block map.
For $a \in \mathcal{A}(\mathbf{X})$, define: $\mathcal{F}_f^{\mathbf{X}}(a) = \{f(aw), aw \sqsubset \mathbf{X}\}$

## Follower-separated

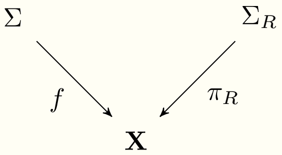
$f : \mathbf{X} \to \mathbf{Y}$ a 1-block map is follower-separated if
$\mathcal{F}_f^{\mathbf{X}}(a) = \mathcal{F}_f^{\mathbf{X}}(b) \Rightarrow a = b$.

## Example: minimal right-resolving cover




$$\pi_R(\Box, q) = \Box$$

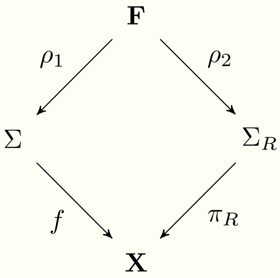
# Current picture




**Hypothesis**

- $f$ onto, right-e-resolving a.e., 1-block
- $\Sigma, \Sigma_R$ 1-step irreducible SFTs
- $\pi_R$ onto, right-resolving, 1-block, follower-separated
- $\mathbf{X}$ irreducible sofic

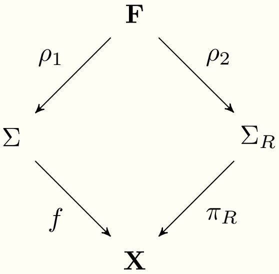
# Fiber product




### Definition

$$\begin{aligned} \mathsf{F} \;=\; & \{(x,y), \\ & x \in \Sigma^{\mathbb{Z}}, y \in \Sigma_R, \\ & f(x) = \pi_R(y)\} \end{aligned}$$

# Fiber product



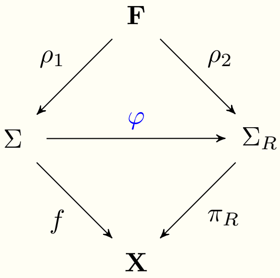

## Definition

$$\mathbf{F} = \{(x,y), \\ x \in \Sigma^{\mathbb{Z}}, y \in \Sigma_R, \\ f(x) = \pi_R(y)\}$$

## Properties

- $\mathbf{F}$ 1-step irreducible SFT
- $\rho_1, \rho_2$ onto
- $\rho_1$ right-resolving, right-e-resolving

# Very useful




## Lemma (Trow'95)

$\exists \varphi : \Sigma^{\mathbb{Z}} \to \Sigma_R, f = \pi_R \circ \varphi \Leftrightarrow \rho_1$ *is bijective.*

## Proof.

Entropy + irreducibility ■

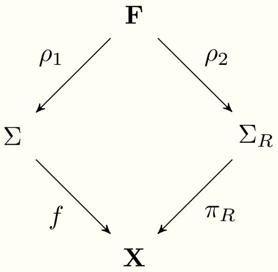
# The proof at least!

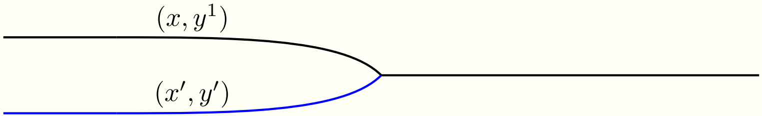
**Lemma**

*$\rho_1$ is bijective.*

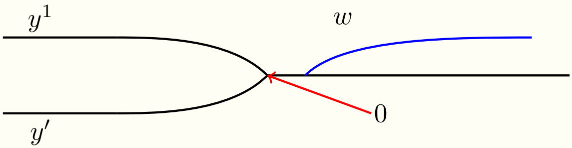
**Proof.**

- Assume not: $(x, y^1), (x, y^2) \in \mathbf{F}$
- $y_0^1 \neq y_0^2$, $\pi_R(y^1) = \pi_R(y^2)$
- Irreducibility: $(x', y')$ left-transitive, $(x', y')_{[0;\infty)} = (x, y^1)_{[0;\infty)}$



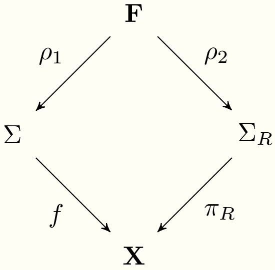



# But it is long...




**Proof.**

- $\pi_R$ follower-separated: $w$, $y^1_0 w \sqsubset \Sigma_R$, $\forall w', \pi_R(w') = \pi_R(w), y^2_0 w' \not\sqsubset \Sigma_R$
- $y'_{(-\infty;1]} w \ldots \in \Sigma_R$



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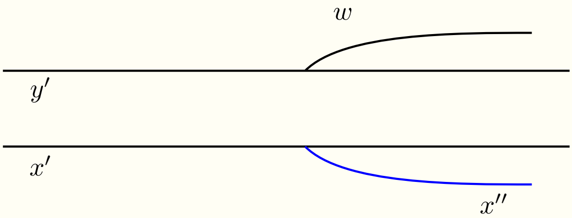
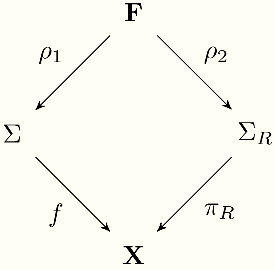
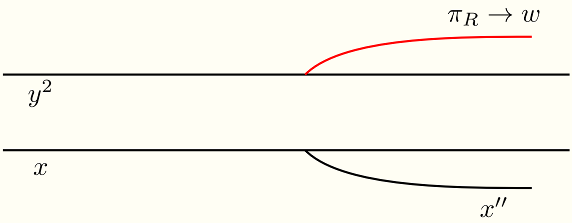



**Proof.**

- $f$ right-e-resolving a.e., $x'$ left-transitive: $\exists x'', f(x'') = \pi_R(y'_{(-\infty;1]} w \dots)$, $x''_{(-\infty;0]} = x'_{(-\infty;0]}$

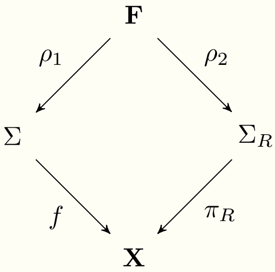


# ...but is ending



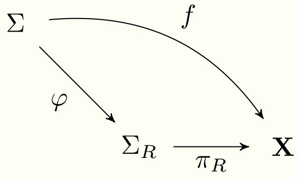

**Proof.**

- $x''_0 = x_0$
- $\rho_1$: right-e-resolving
- $f(x'')_{[1;|w|]} = \pi_R(w)$
- Complete $y^2$: contradiction!



## Theorem

*If $f$ right-e-resolving a.e. then $\exists \varphi : \Sigma \to \Sigma_R$ s.t. $f = \pi_R \circ \varphi$.*



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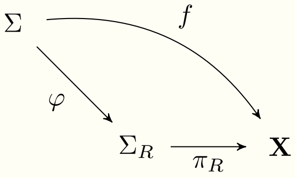



## Going back

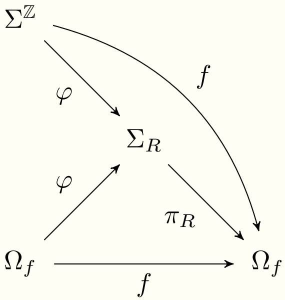
If $f : \mathbf{X} \to \mathbf{X}$, $\varphi(\mathbf{X}) \subseteq \Sigma_R$. Irreducibility + same entropy $\Rightarrow$ $\varphi(\mathbf{X}) = \Sigma_R$

Remember the Boyle-Tuncel result?

**Theorem**

**X** *limit set of a right-continuing almost-everywhere stable CA iff:*

- **X** *factor of a fullshift (trivial necessary condition)*
- $\exists\varphi : \mathbf{X} \to \Sigma_R, \exists\pi : \Sigma_R \to \mathbf{X}$, $\Sigma_R$ *minimal right-resolving cover of* **X**



# Conclusions and questions

## What we did

Characterized stable limit sets of CAs for the cases we know how to construct.

# Conclusions and questions

**What we did**

Characterized stable limit sets of CAs for the cases we know how to construct.

**Question**

Does there exist other types of stable limit sets of CA?