Multidimensional Effective Subshifts

Automata Theory and Symbolic Dynamics Workshop

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ENS de Lyon, CNRS

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In this talk...

- Multidimensional SFT and effective subshifts
- Projective subdynamics
- Implementation of Turing machines inside SFT
- Substitutive subshifts

Outline

Effective subshifts

SFT $\subsetneq$ Sofic susbhifts $\subsetneq$ *Effectively closed*

Definition

A subshift is *effectively closed* (or *effective*) if its complement is a computable union of cylinders.

Property

X is effectively closed if and only one of the followings holds

(i) $X = X_{\mathcal{F}}$ for some recursively enumerable set $\mathcal{F}$ of forbbiden patterns

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Projective subdynamics

Initially introduced by Johnson, Kass and Madden in 2007.

Definition

Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$ subshift and $L \lneq \mathbb{Z}^d$ a k-dimensional sublattice $(1 \leq k < d)$. The *L-projective subdynamics of X* is

$$P_L(X) := \{x|_L : x \in L\} \subseteq \mathcal{A}^L.$$

- $(P_L(X), \sigma_{L \times P_L(X)})$ is a $\mathbb{Z}^k$-subshift.
- $P_L(X)$: globally admissible configurations of shape L in X.
- Loss of information about the original subshift.

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- Loss of information about the original subshift.

In the sequel, we will concentrate on $P_{\vec{e}_1\mathbb{Z}}(X)$ (PS along the horizontal direction).

Subshifts and projective subdynamics

One approach to understand multidimensional SFT is to study their projective subdynamics.

- What are projective subdynamics of 2D sofic subshifts
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  = effective subshifts
- What are projective subdynamics of 2D SFT ?
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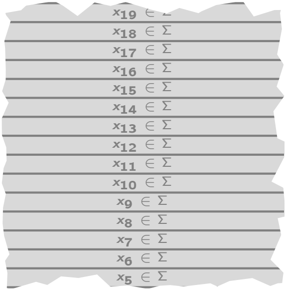
Proposition

Projective subdynamics of SFT (sofic subshifts) are effective subshifts.

What can be PS of sofic subshifts ? (0)

- Trivially, every 1D sofic subshift...

SFT $\Sigma^{\mathbb{Z}}$



$X \subset A^{\mathbb{Z}}$ sofic

$\Sigma \subset B^{\mathbb{Z}}$ SFT, $\Pi : \Sigma \to X$ block map

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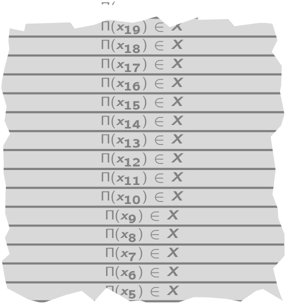

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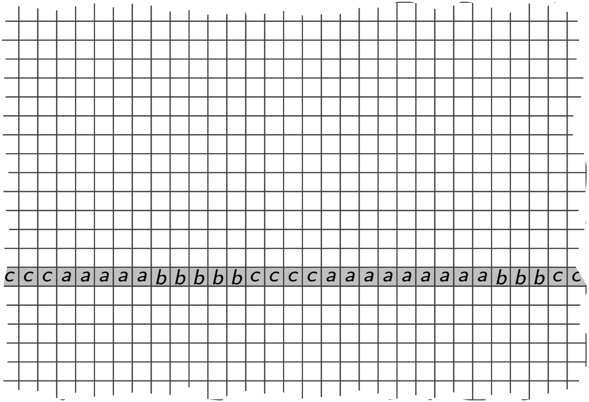

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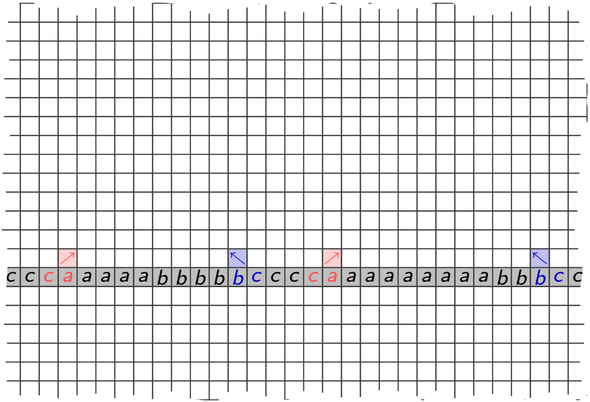

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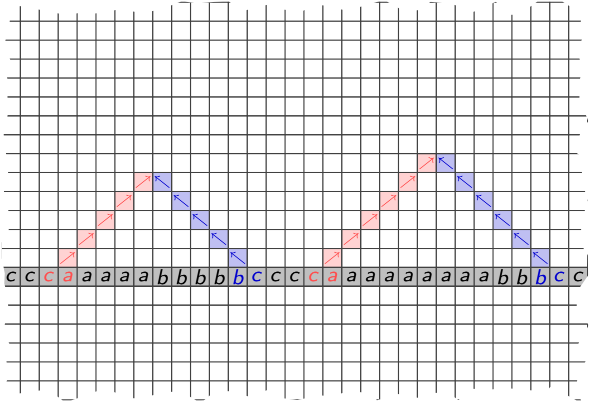

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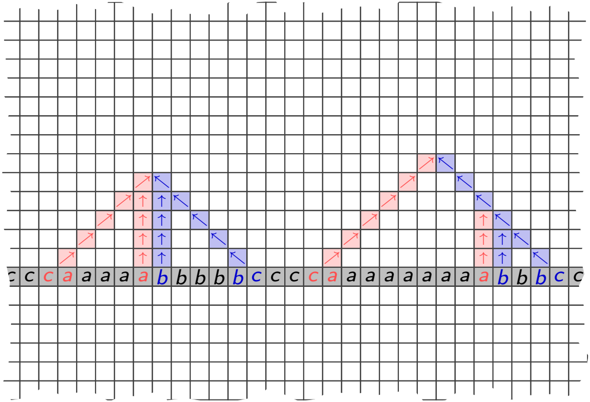

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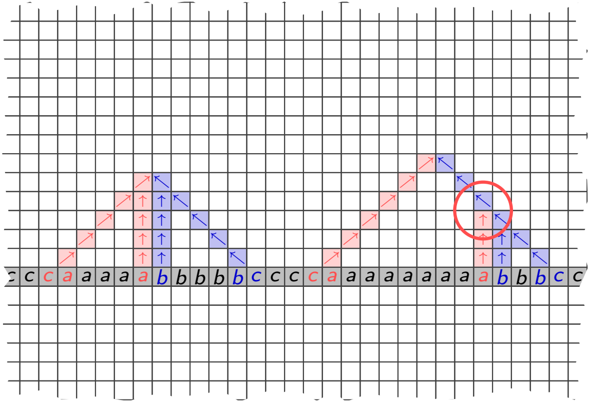

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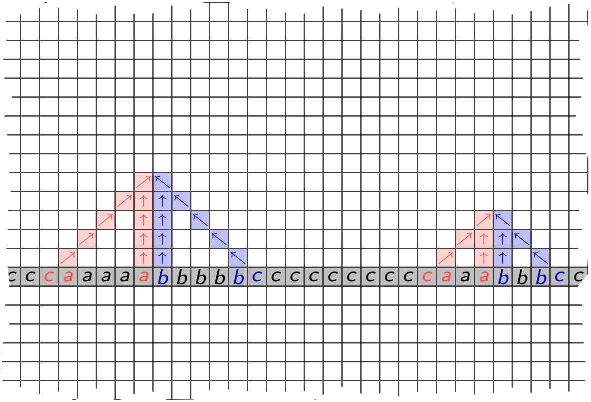

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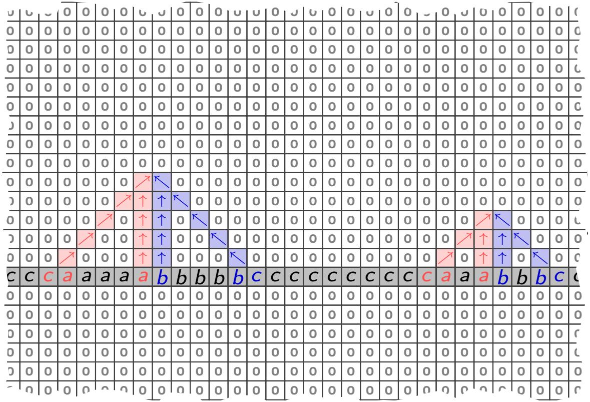
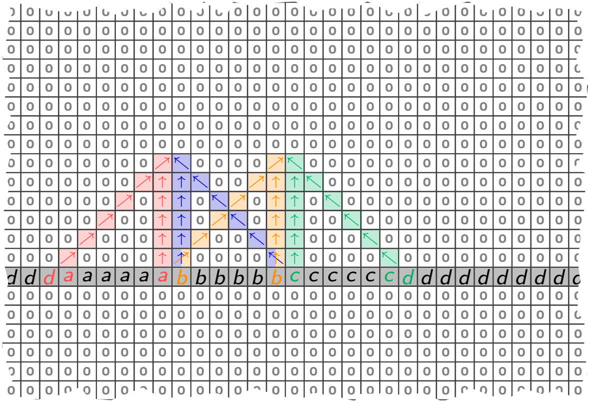


What can be PS of sofic subshifts ? (II)

- The 1D subshift $X_{a^n b^n c^n}$ (neither sofic nor algebraic).



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1. Effective subshifts and projective subdynamics
   - Definition
   - Introductive examples
2. Effective subshifts as projective subdynamics of sofic subshifts
   - Hochman's result
   - Substitutive subshifts
   - Sketch of the proof
3. From $d+2$ to $d+1$
   - A four layers construction
   - Computation stripes
   - Communication channels

Hochman's result

Theorem (Hochman 2008)

Any effective $\mathbb{Z}^d$ subshift may be obtained as the projective subdynamics of a $\mathbb{Z}^{d+2}$ sofic subshift.

The proof is based on

- the use of *Turing machines as SFT*,
- *substitutive tilings* to construct computation zones in 3D.

Substitutive subshifts

We consider only *rectangular substitutions* on a finite alphabet A.

If s is such a substitution, the *s-patterns* are the $s^n(a)$ for every letter a and every integer $n \in \mathbb{N}$ (if they are well-defined).

Definition

Let s be a rectangular substitution on A. Then the *substitutive subshift generated by s* is

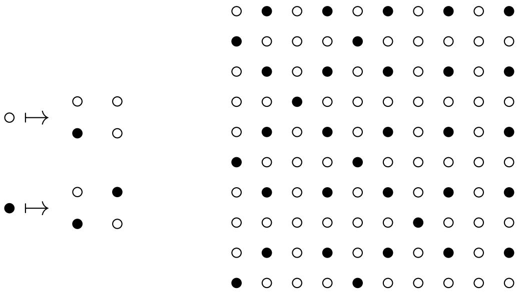
$$X_s = \left\{ x \in A^{\mathbb{Z}^2} : \text{ every pattern of } x \text{ is a } s\text{-pattern} \right\}.$$

Mozes' Theorem

Theorem (Mozes, 1989)

If the substitution s has *good properties* (for instance deterministic), then the subshift X_s is sofic.

Idea of the proof for 2×2 substitutions



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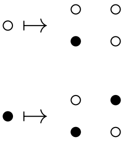

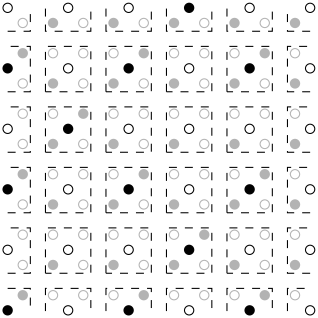

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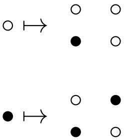
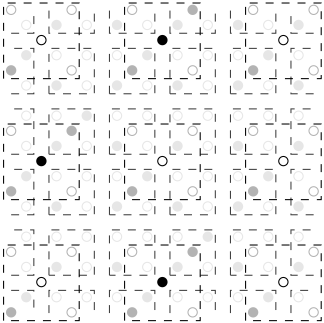




Hochman's proof: a 3D construction

Start with two rectangular substitutions s_3 and s_5

$$s_3 \quad \bullet \mapsto \begin{matrix} \circ & \circ & \bullet \\ \circ & \bullet & \circ \\ \bullet & \circ & \circ \end{matrix}, \quad \circ \mapsto \begin{matrix} \circ & \circ & \circ \\ \circ & \bullet & \circ \\ \bullet & \circ & \circ \end{matrix}$$

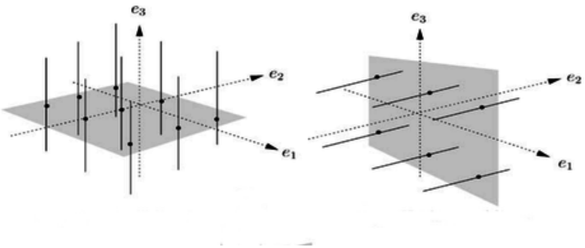
$$s_5 \quad \bullet \mapsto \begin{matrix} \circ & \circ & \circ & \circ & \bullet \\ \circ & \circ & \circ & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \bullet & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ & \circ \end{matrix}, \quad \circ \mapsto \begin{matrix} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \bullet & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ & \circ \end{matrix}$$

Mozes' result $\Rightarrow$ 2D *sofic subshifts* W_3 and W_5.

Hochman's proof: a 3D construction

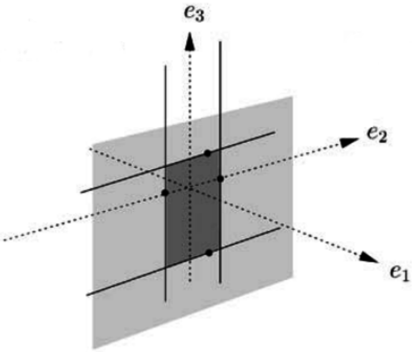
Identical copies of W_3 along direction $\vec{e_3}$ and of W_5 along $\vec{e_2}$

- Copies of W_3 produce *vertical lines*
- Copies of W_3 produce *horizontal lines*



Hochman's proof: a 3D construction

Thus some rectangles appear !



And all rectangles are the same on one plane.

Hochman's proof: a 3D construction

These rectangles have good properties

- there are only finitely many planes with infinite rectangles
- each set $[k, k+n]\vec{e_2}$ will appear in arbitrarily large rectangles

Thus if $\mathcal{M}$ is a TM that enumerates F

- we can put calculations of $\mathcal{M}$ (real time Turing machine) in each rectangle
- each time a forbidden pattern is produced, its presence is checked inside the rectangle
- rectangles repartition $\Rightarrow \mathbb{Z}\vec{e_2}$ is entirely scanned

$\Rightarrow$ The subshift X_F exactly appears on $\mathbb{Z}\vec{e_2}$.

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From $d + 2$ to $d + 1$

Hochman's result for effective subshifts can be made *optimal* in terms of dimension.
(since there exist non-sofic effective subshifts, dimension d is impossible)

Theorem (Durand, Romaschenko & Shen 2011, A.& Sablik 2013)

Any effective $\mathbb{Z}^d$-subshift may be obtained as the projective subdynamics of a $\mathbb{Z}^{d+1}$ sofic subshift.

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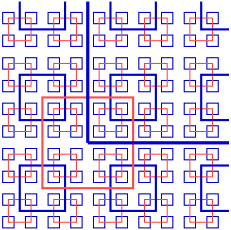
Two independent proofs

- the first one is based on *self-similar tilings*
- the second one uses *Robinson like* techniques

From $d+2$ to $d+1$: Sketch of the proof

What about Robinson tiling ?



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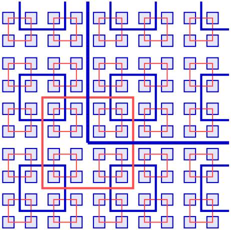

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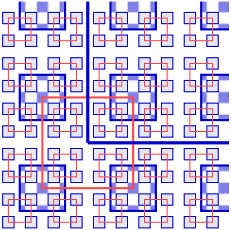

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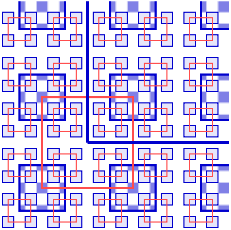

But...

- Computation zones are squares !
- How to solve the *disconnected tape* problem ?

A four layers construction

How to realize an effective 1D-subshift $\Sigma \subset {\mathcal{A}_\Sigma}^{\mathbb{Z}}$ as PS of a 2D sofic subshift ?

- SFT made of four layers
  - first layer: configuration $x \in \mathcal{A}_\Sigma^{\mathbb{Z}}$ that will be checked
  - second layer: hierarchical structure: computation zones for TM
  - third layer: TM $\mathcal{M}_{\mathtt{F}}$ that enumerates forbidden patterns of Σ and checks if $x \in \Sigma$
  - fourth layer: TM $\mathcal{M}_{\mathtt{Search}}$ that helps the TM $\mathcal{M}_{\mathtt{F}}$ to scan entirely x
- all layers but the first are finally erased with a letter-to-letter block map

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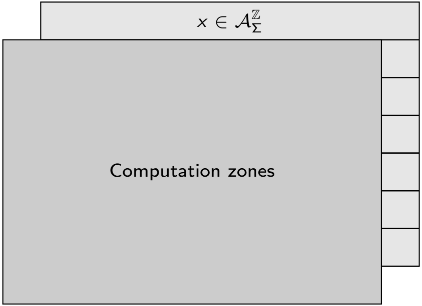

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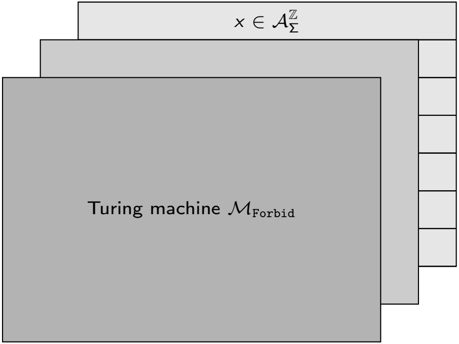

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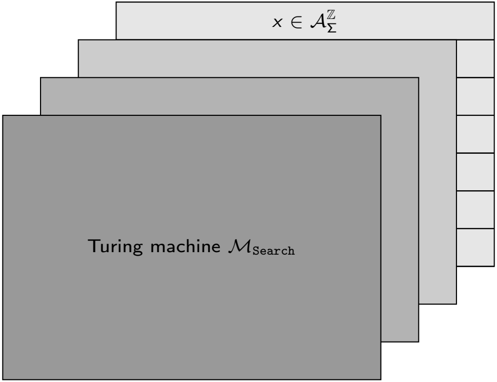

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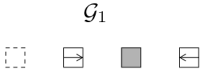
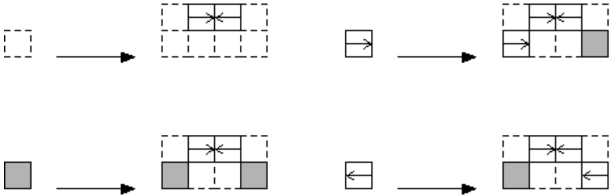
Layer 2: Computation zones

Alphabet $\mathcal{G}_1$



Substitution rules of $s_{\texttt{Grid}}$:



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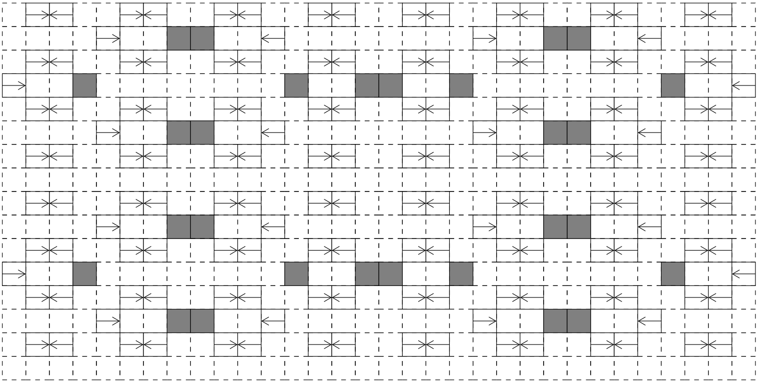
After some iterations. . .



□ : communication tile

⇥, ⇤, ■ : computation tiles

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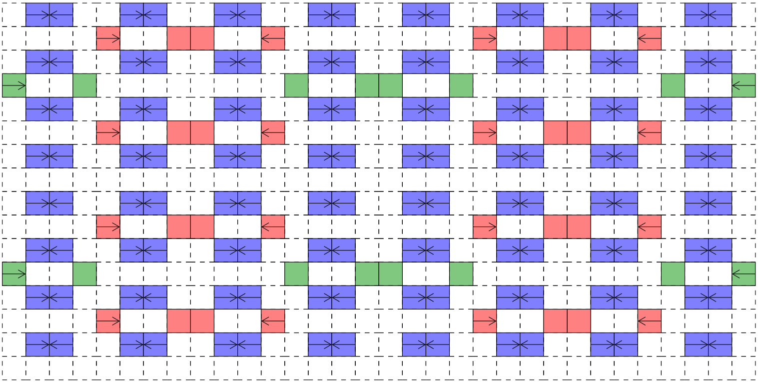

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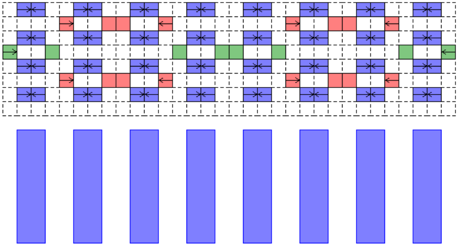
Layer 2: Computation zones

Stripes of different levels (level 1, level 2, level 3):



A stripe of level n has the following properties

- width 2^n,
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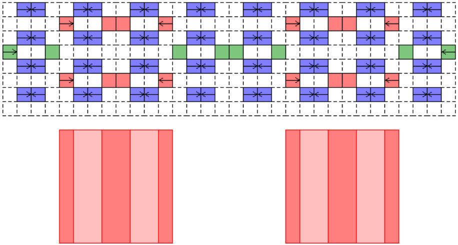

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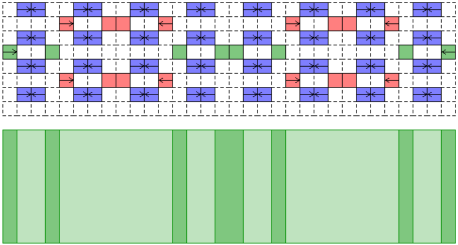

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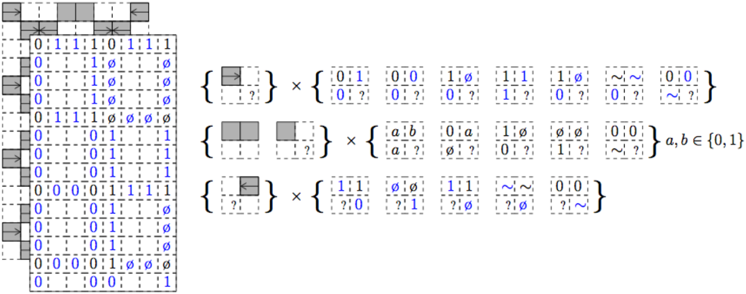
Layer 2: the clock

To initialize calculations we code a clock by local rules



In a level n stripe, calculations are initialized every 2^{2^n} steps of calculation.

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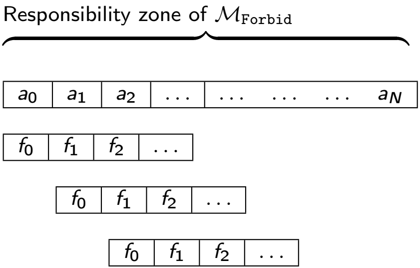
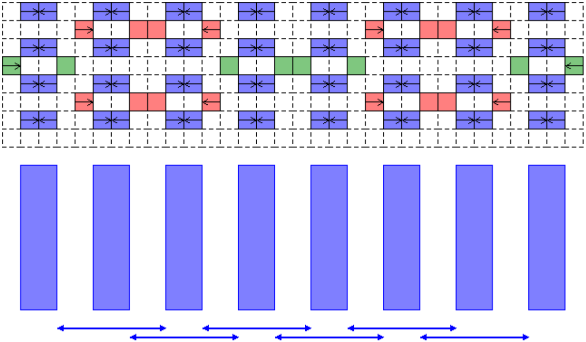
- $\mathcal{M}_{\texttt{Forbid}}$ generates forbidden patterns of Σ
- each stripe has a *responsibility zone* and $\mathcal{M}_{\texttt{Forbid}}$ verifies that no forbidden pattern appears inside this zone;



- to get symbol a_k from level 1, $\mathcal{M}_{\texttt{Forbid}}$ is helped by $\mathcal{M}_{\texttt{Search}}$: $\mathcal{M}_{\texttt{Forbid}}$ gives the address k and gets a_k.

Responsibility zone of $\mathcal{M}_{\texttt{Forbid}}$

Responsibility zones must overlap



A Turing machine $\mathcal{M}_{\texttt{Forbid}}$ of level n may ask help from a $\mathcal{M}_{\texttt{Search}}$ of same level or an adjacent $\mathcal{M}_{\texttt{Search}}$ of same level.

Responsibility zone of $\mathcal{M}_{\texttt{Forbid}}$

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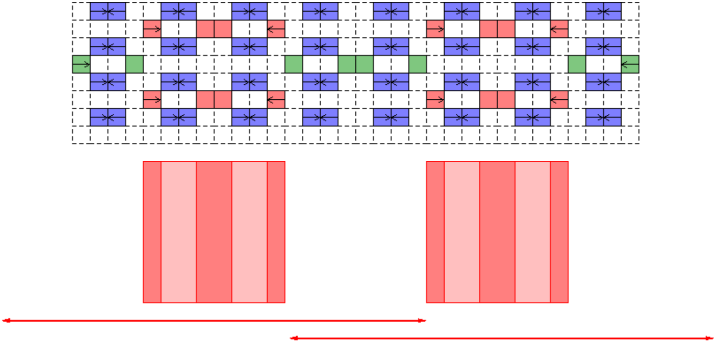

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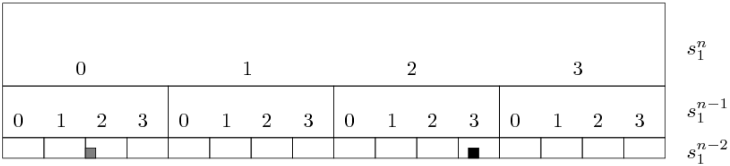
Layer 4 : Turing machine $\mathcal{M}_{\texttt{Search}}$

A $\mathcal{M}_{\texttt{Search}}$ machine of level n can communicate with $\mathcal{M}_{\texttt{Search}}$ machines of levels $n-1$ and $n+1$.

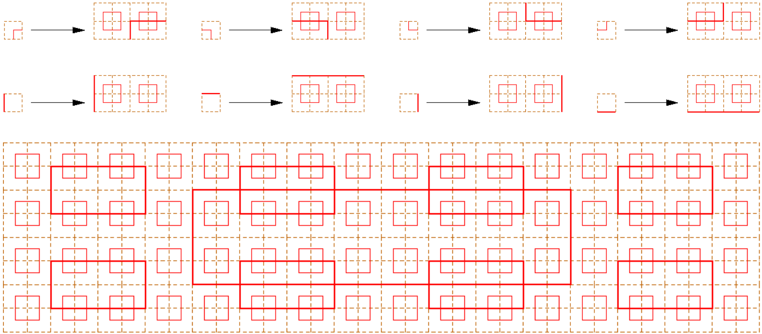
Given a computation stripe of level n, each symbol is given an address, and this address is compatible with addresses of levels $n-1$ and $n+1$.



The address of ■ is 231 and the address of ■ is 020.

Communication between $\mathcal{M}_{\texttt{Search}}$ of different levels

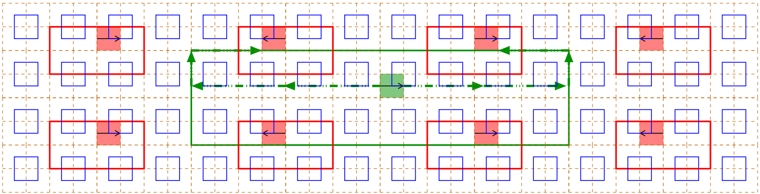
With a new alphabet $\mathcal{G}_2$, we construct *communication channels*



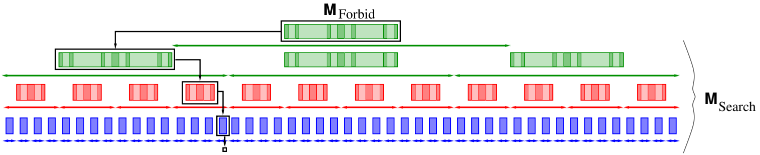
Communication between $\mathcal{M}_{\texttt{Search}}$ of different levels

Communication channels are such that

- every tile $\boxed{\rightarrow}$ or $\boxed{\leftarrow}$ is in the center of a rectangle of level n;
- every rectangle of level n is connected to the $\boxed{\rightarrow}$ and $\boxed{\leftarrow}$ of two stripes of level $n-1$



$\mathcal{M}_{\texttt{Search}}$ works !



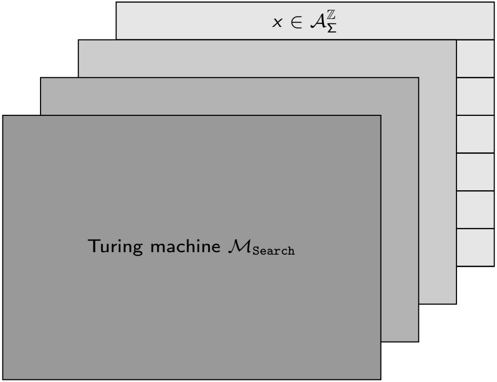
The machines $\mathcal{M}_{\texttt{Search}}$ work as we expect:

- every $\mathcal{M}_{\texttt{Search}}$ has enough space to code addresses
- every $\mathcal{M}_{\texttt{Search}}$ has enough time to perform calculations (exponential clock)

A four layers construction

How to realize an effective 1D-subshift $\Sigma \subset \mathcal{A}_\Sigma^{\mathbb{Z}}$ as PS of a 2D sofic subshift ?

- SFT made of four layers
  - first layer: configuration $x \in \mathcal{A}_\Sigma^{\mathbb{Z}}$ that will be checked
  - second layer: hierarchical structure: computation zones for TM
  - third layer: TM $\mathcal{M}_{\mathtt{F}}$ that enumerates forbidden patterns of Σ and checks if $x \in \Sigma$
  - fourth layer: TM $\mathcal{M}_{\mathtt{Search}}$ that helps the TM $\mathcal{M}_{\mathtt{F}}$ to scan entirely x
- all layers but the first are finally erased with a letter-to-letter block map



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$x \in \Sigma$
$x \in \Sigma$
$x \in \Sigma$
$x \in \Sigma$
$x \in \Sigma$
$x \in \Sigma$
$x \in \Sigma$

Some applications

- Characterization of possible entropies of 2D SFT [Hochman & Meyerovitch, 2010]
- Multidimensional effective S-adic subshifts are sofic [A. & Sablik, submitted]
- There exists a sofic subshift whose quasi-periodic configurations have a non-recursively bounded periodicity function [Ballier & Jeandel, 2010]
- A computable planar tiling admits local rules [Fernique & Sablik, 2012]

Improvement, Limitation and Question

- Is it possible to determinize the construction (deterministic SFT) ?
  $\rightsquigarrow$ It should be. . . [Guillon & Zinoviadis, in progress]

- The construction is highly constrained, in the sense that the sofic subshift is constant along the vertical direction ($\Rightarrow$ zero entropy).
  $\rightsquigarrow$ What are PS of mixing sofic subshifts/SFT ?

Conclusion

- 1D effective subshifts as PS of 2D sofic subshifts...
- ...but very constrained construction (zero entropy) !
- Another approach: impose that lines are in some subshift X_H, what subshift X_V can you get on the columns ?

Conclusion

- 1D effective subshifts as PS of 2D sofic subshifts...
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Thank you for your attention !