Explicit zero density for the Riemann zeta function

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Overview I

Introduction and Motivation

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Definition ($\zeta(s)$)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\sigma > 1$, and for the remainder of the complex plane, it is defined as the analytic continuation of the above function.

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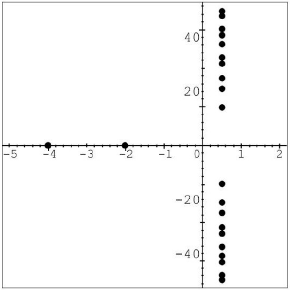
The study of zeros of the zeta function plays an important role in analytical number theory. Riemann Hypothesis(RH) is about the locations of zeros of Riemann zeta function. According to this hypothesis $\zeta(s)$ has trivial zeros at negative even integers, that is $s = -2, -4, -6, ...,$ and no other real zeros and non-real zeros, also called the nontrivial zeros, are lie on the critical line $R(s) = \frac{1}{2}$, and to date still is an open problem.



Motivation

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$$\pi(x) = \mathrm{Li}(x) + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right)$$

For some $c > 0$, where $\pi(x)$ is the number of primes less than or equal to x and $\mathrm{Li}(x)$ is the logarithmic integral function defined by $\mathrm{Li}(x) = \int_2^x \frac{1}{\log t} dt$. The shape and constant in the error term are determined by what we can prove about the number and location of zeros of zeta function.

Zero density

Definition

Let $\frac{1}{2} < \sigma < 1$, $T > 0$, we have

$$N(\sigma, T) = \#\{\rho = \beta + i\gamma : \zeta(\rho) = 0, 0 < \gamma < T, \sigma < \beta < 1\}. \qquad (1)$$

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$$N(\sigma, T) = \#\{\rho = \beta + i\gamma : \zeta(\rho) = 0, 0 < \gamma < T, \sigma < \beta < 1\}. \quad (1)$$

We want to find an explicit upper bound for the number of zeros of the zeta function within this rectangular region. This type of result is commonly referred to as a **zero density** result.

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As T grows to infinity. In 1937 Ingham[2] showed that

$$N(\sigma, T) = O\left(T^{(2+4c)(1-\sigma)}(\log T)^5\right).$$

By assuming that $\zeta(\frac{1}{2} + it) = O(t^{c+\epsilon})$.

History

Ramaré[3] had proven an explicit version of Ingham's bound. For example, for $\sigma = 0.90$ this formula simplifies to

$$N(0.90, T) < 1293.48(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 51.50(\log T)^2.$$

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If we put $\sigma = 0.90$, We can see how this improves on Ramare's estimate.

$$N(0.90, T) < 11.499(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 3.186(\log T)^2.$$

Main goal of my thesis

Goal:

- Find better values of $C_1(\sigma)$ and $C_2(\sigma)$.
- Update the result to use better bounds on ζ on the half line which will improve the exponents on both the T and $\log T$ terms.
- Improve the bounds for some arithmetic sums which effects on the $C_2(\sigma)$ term.

General Methods

Counting zeros of a function in a rectangle region

There exists many useful tools in complex analysis to count the zeros of the holomorphic function inside a specified rectangle region and, in my thesis, by using the classic idea of Bohr, Landau, Littlewood and Titchmarsh as stated in [5] which uses the residue theorem, we can bound the number of zeros of a function in a rectangle region. Specifically, we will try to bound the number of zeros of a function h as in:

$$N_h(\sigma, T_1, T_2) = \# \left\{ \rho' = \beta' + i\gamma'; h\left(\rho'\right) = 0, \sigma < \beta' < 1, T_1 < \gamma' < T_2 \right\}. \tag{2}$$

Counting zeros of a function in a rectangle region

First, we compare the number of zeros for our function to its average:

$$N_h(\sigma, T_1, T_2) \leq \frac{1}{\sigma - \sigma'} \int_{\sigma'}^{\mu} (N_h(\tau, T_1, T_2)) d\tau. \tag{3}$$

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Using the residue theorem in complex analysis, we have

$$\frac{-1}{2\pi i} \int_R \log h(s) ds = \int_{\sigma'}^{\mu} N(\tau, T_1, T_2) d\tau. \tag{4}$$

where R is the boundary of the region.

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where R is the boundary of the region. Therefore, we have

$$N_h(\sigma, T_1, T_2) \leq \frac{1}{2\pi(\sigma - \sigma')} \Re \int_R \log h(s) ds. \tag{5}$$

Counting zeros of a function in a rectangle region

Theorem (E. Titchmarsh, 2011)

Let $h(s)$ is a meromorphic function in and upon the boundary of a rectangle bounded by the lines $t = T_1, t = T_2, \sigma = \sigma'$ and $\sigma = \mu$. $(\mu > \sigma')$ and suppose that $N(\sigma, T_1, T_2)$ is the number of zeros of the function $h(s)$. Then for counting zeros of the function in that specified rectangle region, we use the below inequality,

$$N_h(\sigma, T_1, T_2) \leq \frac{1}{2\pi(\sigma - \sigma')}\left(\int_{T_1}^{T_2} \log\left|h\left(\sigma' + it\right)\right|dt + \int_{\sigma'}^{\mu} \arg h(\tau + iT)d\tau - \int_{\sigma'}^{\mu} \arg h(\tau + iH)d\tau - \int_{T_1}^{T_2} \log|h(\mu + it)|dt\right). \tag{6}$$

Goal of the Analysis

Goal: To find an **upper bound** for each integral in the expression for $N_h(\sigma, T_1, T_2)$.

Key Insight:

We note that in the application to $\zeta(s)$, as T_2 grows larger, the main contribution comes from the **first integral**, $\int_{T_1}^{T_2} \log|h(\sigma' + it)|dt$.

Challenges and Approach

Challenges in Bounding the first integral, $\int_{T_1}^{T_2} \log \left| h\left(\sigma' + it\right) \right| dt$

While estimating the integral of $\log h(\sigma' + it)$ two key challenges arise:

- logarithmic functions are not analytic.

Challenges in Bounding the first integral, $\int_{T_1}^{T_2} \log\left|h\left(\sigma' + it\right)\right| dt$

While estimating the integral of $\log h(\sigma' + it)$ two key challenges arise:

- logarithmic functions are not analytic.
- We don't have good understanding of zeta function in the critical strip.

Our Solutions to Overcome the Challenges

To address these challenges, our first solution is:

- Instead of finding an upper bound for integral of $\log h(\sigma' + it)$, we can find an upper bound for the integral of $h(\sigma' + it)$, since we have the following inequality:

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$$\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \log(|h(\sigma' + it)|)dt \le \log\left(\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} |h(\sigma' + it)|dt\right). \tag{7}$$

Our Solutions to Overcome the Challenges

Second solution is applying the convexity estimate:

- It basically says that the integral of the form $J(\sigma') = \int_{-\infty}^{\infty} |h(\sigma' + it)| dt$ is *log*-convex, so that the value in the middle is controlled by the integral on the sides, where h is an analytic complex function.

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$$\boldsymbol{J(\sigma) \leq (J(\sigma_1))^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}} \cdot (J(\sigma_2))^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}}}$$

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For instance in our context, we have good information about zeta function outside of the critical strip, and, reasonable information at $\frac{1}{2}$ line. Therefore, we use the convexity bound for integrals to move the problem to each of the two boundaries.

But what is the problem of this method?

Problem

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convexity estimate creates another problem that the integral is of the from

$$\int_{-\infty}^{\infty} |h(\sigma' + it)| dt.$$

Solution

Therefore, our solution is for using the convexity estimate we first use the smoothing method to smooth the function at σ' and then for getting bounds at the boundaries we use unsmoothing method to turn problem back to the original integrals at each of the boundaries.

Smoothing method

The general idea of this method is to replace a bounded sum or integral of a function with an infinite sum or integral of a smoothed version of the function. To obtain this, we introduce a smooth weight function g as a characteristic function.

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Definition

Let $\alpha > 0$. We have

$$g(s) = \frac{s-1}{s} e^{\alpha \left(\frac{s}{T}\right)^2}.$$

Smoothing method

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Therefore, we have

$$\int_{T_1}^{T_2} |h(\sigma+it)|dt \leq \frac{\int_{-\infty}^{\infty} |g(\sigma+it)||h(\sigma+it)|dt}{\omega_2},$$

where ω_2 is a positive number depend on g.

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where ω_2 is a positive number depend on g.We denote

$$J(\sigma) = \int_{-\infty}^{\infty} |g(\sigma + it)||h(\sigma + it)| dt. \tag{8}$$

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$$J(\sigma) \leq \omega_1 \int_0^{\infty} x^{\beta-1} e^{-2\alpha x^{\beta}} F(\sigma, xT)dx$$

Where ω_1 is a positive function depend on g and

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Therefore, by using unsmoothing method we can find an upper bound for $J(\sigma_1)$ and $J(\sigma_2)$.
To obtain these two bounds, we need to find a bound for $F(\sigma, xT)$ at σ_1 and σ_2.

Main Theorem

Theorem (G.Farzanfard)

Suppose that there exist a, a_i, b, b_i, d, d_i depending on σ_1 and $a', a_i', b', b_i', d', d_i'$ depending on σ_2 such that, for all t we have

$$\begin{aligned} F(\sigma_1, t) &\le dt^a (\log t)^b \sum_i d_i t^{a_i} (\log t)^{b_i} \\ F(\sigma_2, t) &\le d' t^{a'} (\log t)^{b'} \sum_{i'} d_i' t^{a_i'} (\log t)^{b_i'}, \end{aligned} \tag{9}$$

where $\sum_i d_i t^{a_i} (\log t)^{b_i} = 1 + o(1)$ and $\sum_{i'} d_i' t^{a_i'} (\log t)^{b_i'} = 1 + o(1)$. Then

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$$F(\sigma_1, t) \le dt^a (\log t)^b \sum_i d_i t^{a_i} (\log t)^{b_i}$$
$$F(\sigma_2, t) \le d' t^{a'} (\log t)^{b'} \sum_{i'} d'_i t^{a'_i} (\log t)^{b'_i}, \tag{9}$$

where $\sum_i d_i t^{a_i} (\log t)^{b_i} = 1 + o(1)$ and $\sum_{i'} d'_i t^{a'_i} (\log t)^{b'_i} = 1 + o(1)$. Then

$$F(\sigma, T_1, T_2) \le C_5 \, T_2^{a\left(\frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1}\right) + a'\left(\frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}\right)} \log T_2^{b\left(\frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1}\right) + b'\left(\frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}\right)}, \tag{10}$$

where $C_5 = C_5(\sigma, T_1, T_2, a_i, a'_i, b_i, b'_i, d_i, d'_i)$ is decreasing as $T_2 \to \infty$.

Challenges in Bounding the fourth integral, $-\int_{T_1}^{T_2} \log|h(\mu+it)|dt$

For finding an upper bound for the integral, $-\int_{T_1}^{T_2} \log|h(\mu+it)|dt$, we need to find an explicit lower bound for $\int_{T_1}^{T_2} \log|h(\mu+it)|dt$. In our context, we have the following challenge:

- Finding the lower bound on the integral of the zeta function outside of the critical strip is challenging.

Our Solutions to Overcome the Challenges

To address this challenge, we use mollifiers.

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Definition of Mollifier

Let $X \geq 1$ be a parameter. We define the mollifier as follows:

$$M_X(s) = \sum_{n \leq X} \frac{\mu(n)}{n^s}, \tag{11}$$

where $\mu(n)$ is the Möbius function.

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Let $f(s) = \zeta(s) \cdot M_X(s) - 1$. The series expansion for $f(s)$ is expressed as a below dirichlet series

$$f(s) = \sum_{n \geq 1} \frac{\lambda(n)}{n^s},$$

$$\text{with} \begin{cases} \lambda_x(n) = 0 \text{ if } n \leq x \\ \lambda_x(n) = \sum_{\substack{d|n \\ d \leq x}} \mu(d) \text{ if } n > x \end{cases}.$$

Advantage of the function h

Remark

The function $h_X(s)$ has an advantage: outside the critical strip, it is close to 1, making it easier to estimate accurately. Furthermore, compared to the product $M_X(s)\zeta(s)$, new function $h_X(s)$ involves second moments in certain estimates, which allows us to use the mean value theorem.

Application

Application: First integral

Now, to apply the general methods in our **Main Theorem**, we need to bound:

- $F\left(\frac{1}{2}, T\right) = \int_0^T |f(\frac{1}{2} + iT)|^2 dt$,
- $F\left(1 + \frac{\delta}{\log X}, T\right) = \int_0^T |f(1 + \frac{\delta}{\log X} + iT)|^2 dt$.

Remark

In order to bound the first integral, we will introduce the specific bounds required for $F(\sigma, T)$ to proceed with the application of the Main Theorem.

Bounding $F(\frac{1}{2}, T)$

To bound $F(\frac{1}{2}, T)$, we first need to bound

$$\int_0^T |\zeta(\frac{1}{2} + it)^2 M_X(\frac{1}{2} + it)^2| dt. \tag{12}$$

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This integral is bounded by

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- Bounds for $\zeta(s)^2$ on the critical line.
- bound for the integral, $\int_0^T \left|M_X(\frac{1}{2} + it)\right|^2 dt$.

Bounding $|\zeta(\frac{1}{2}+it)|^2$

Kadiri, Lumely and Ng used the below bound,

$$\max_{|t|\leq T} |\zeta(\frac{1}{2}+it)|^2 \leq (0.63\,T^{\frac{1}{6}}\log T + 2.851)^2, \qquad T > 0$$

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To improve the final bound, we will substitute improved bounds for $\zeta(s)$. Hiary, Patel and Yang [6] improved the lead constant for $T > 3$, we use this to prove:

$$\max_{|t|\leq T} |\zeta(\frac{1}{2}+it)|^2 \leq (0.618)^2 T^{\frac{1}{3}} \log^2 T + \zeta(\frac{1}{2})^2, \qquad T > 0$$

Patel and Yang [7], removed the log term and so we obtain:

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Patel and Yang [7], removed the log term and so we obtain:

$$\max_{|t|\leq T} |\zeta(\frac{1}{2} + it)|^2 \leq (66.7)^2 t^{\frac{27}{82}} + \zeta(\frac{1}{2})^2, \qquad T > 0$$

bounding $\int_0^T \left|M_X(\frac{1}{2}+it)\right|^2 dt$

We use Montgomery and Vaughan's mean value theorem for the sums involving real-valued sequences:

$$\int_0^T \left|\sum_{n=1}^{\infty} u_n n^{it}\right|^2 dt \leq \sum_{n\geq 1} |u_n|^2 \left(T + \pi m_0(n+1)\right).$$

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For a real-valued sequence u_n. we apply this to $M_X(s) = \sum_{n\leq X} \frac{\mu(n)}{n^s}$. Letting $u_n = \frac{\mu(n)}{n^{\frac{1}{2}}}$, we get:

$$\int_0^T M_X(\frac{1}{2}+it)^2 dt \leq (T + \pi m_0)) \sum_{n\leq X} \frac{\mu^2(n)}{n} + m_0\pi \sum_{n\leq X} \mu^2(n)$$

bounds on $\sum_{n\leq X}\mu^2(n)$ and $\sum_{n\leq X}\frac{\mu^2(n)}{n}$

Here, we have bounds provided by Ramare. For each $X \geq 1700$, we have

$$\sum_{n\leq X}\mu^2(n) \leq 0.62X \tag{14}$$

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Here, we have bounds provided by Ramare. For each $X \geq 1700$, we have

$$\sum_{n\leq X}\mu^2(n) \leq 0.62X \tag{14}$$

and for $X \geq 1000$, we have

$$\sum_{n\leq X}\frac{\mu^2(n)}{n} \leq \frac{6}{\pi^2}\log X + 1.048. \tag{15}$$

bounds on $\sum_{n\le X}\mu^2(n)$ and $\sum_{n\le X}\frac{\mu^2(n)}{n}$

Here, we have bounds provided by Ramare. For each $X \geq 1700$, we have

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and for $X \geq 1000$, we have

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By combining these two bounds and simplification, we obtain below bound:

$$\boxed{\int_0^T |M_X(\tfrac{1}{2}+it)|^2 dt \le (C_1 T + C_2 X)\log X.}$$

Bounds for $F(\frac{1}{2}, T)$

Now with the combination of the bounds of the integral $\int_0^T \left|M(\frac{1}{2}+it)\right|^2 dt$ and different bounds with different shapes for zeta function on the half line, we derive the following estimates for $F(\frac{1}{2}, T)$:

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Remark

In our thesis we will, at this stage, keep lower order terms.

Bounding $F(1 + \frac{\delta}{\log X}, T)$

Based on the expansion of f, We can write,

$$F(\sigma_2, T) = \int_0^T |f(\sigma_2 + it)|^2 \, dt = \int_0^T \left| \sum_{n \geq 1} \frac{\lambda(n)}{n^{\sigma_2 + it}} \right|^2 dt.$$

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Letting $u_n = \frac{\lambda_X(n)}{n^{\sigma_2}}$ and using the Montgomery and Vaughan's mean value theorem, we obtain below bound:

$$\pi m_0 \sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2\sigma_2 - 1}} + (T + \pi m_0) \sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2\sigma_2}}.$$

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For finding the upper bound for the above summations, we rewrite the bounds provided by Ramare but improved the constants.

Bounds on $\sum_{n\geq 1}\frac{\lambda_X(n)^2}{n^{2\sigma_2-1}}$ and $\sum_{n\geq 1}\frac{\lambda_X(n)^2}{n^{2\sigma_2}}$

Ramare provided below bounds:

$$\sum_{n\geq 1}\frac{\lambda_X(n)^2}{n^{1+2\frac{\delta}{\log X}}}\leq \frac{b_4}{2\delta}\left(1+\frac{2\delta}{\log X}\right)^2 e^{\frac{2\delta\gamma}{\log X}}(\log X)^2, \tag{16}$$

and

$$\sum_{n\geq 1}\frac{\lambda_X(n)^2}{n^{2+\frac{2\delta}{\log x}}}\leq \frac{b_4}{5\delta e^{\delta}}\left(1+\frac{\delta}{\log X}\right)^2\frac{(\log X)^2}{X}+\frac{b_3e^{-2\delta}}{X}. \tag{17}$$

where $b_4=0.529$.

Bounds on $\sum_{n\geq 1} \frac{\lambda_X(n)^2}{n^{2\sigma_2 - 1}}$ and $\sum_{n\geq 1} \frac{\lambda_X(n)^2}{n^{2\sigma_2}}$

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Our Contribution:

A.Fiori and G.Farzanfard improved the bounds by improving the constant b_4 to 0.470 and removing the $\frac{\delta}{\log X}$.

Bound on $F(1 + \frac{\delta}{\log X}, T)$

By combining these two obtained bounds, we have:

$$F(1 + \frac{\delta}{\log X}, T) \leq \left(D_2 + \frac{T}{X} D_2\right) (\log X)^2 . \tag{18}$$

Bounds on $F(\sigma, T)$

By the obtained bounds for $F_X(\frac{1}{2}, T)$ and $F_X(1 + \frac{\delta}{\log X}, T)$ we can apply our main theorem and provide two different versions of a bound for $F(\sigma, T)$.

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$$F(\sigma, T) \le D_3 T^{\frac{4}{3}(\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})} \log T^{3(\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})+2(\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})}, \tag{19}$$

$$F(\sigma, T) \le D_3' T^{\frac{109}{82}(\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})} \log T^{2(\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})}. \tag{20}$$

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Remark

This gives us a bound for the first integral.

Fourth Integral: $-\int_{T_1}^{T_2} \log|h(\mu+it)dt|$

The fourth bound we need is an upper bound for $-\int_{T_1}^{T_2} \log|h(\mu+it)dt|$ or equivalently an explicit lower bound for $\int_{T_1}^{T_2} \log|h(\mu+it)|dt$. This is very similar to bounding $F(\sigma_2, T)$ that we had in the first integral.

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Remark: $\mu \geq 1.23622...$, $|f(\mu + it)| < 1$.

$$-\log |1 - f(\mu + it)^2| \leq -\log(1 - |f(\mu + it)|^2) \leq b_1 |f(\mu + it)|^2,$$

where b_1 depends on the lower bound for f.

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where b_1 depends on the lower bound for f.

So by using the Montgomery and Vaughan's mean value theorem, we have the below bound for the last integral

$$-\int_{T_1}^{T_2} \log|h_X(\mu+it)|dt \leq \frac{A(logX)^3}{X^{2\mu-2}} + \frac{B(logX)^3 T}{X^{2\mu-1}}.$$

$\int_{\sigma'}^{\mu} \arg h(\tau + iT_2) d\tau - \int_{\sigma'}^{\mu} \arg h(\tau + iT_1) d\tau.$

We are looking to bound the difference of two integrals that involve the argument of a holomorphic function:

$$\int_{\sigma'}^{\mu} \arg h(\tau + iT_2) d\tau - \int_{6'}^{\mu} \arg h(\tau + iT_1) d\tau.$$

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which is

$$\leq (\mu - \sigma') \max_{\tau \in (\sigma', \mu)} (|\arg h(\tau + iT_2)| + |\arg h(\tau + iT_1)|).$$

General Idea of bounding the $|argh(\tau + iU)|$

Kadiri, Lumely and Ng proved a bound that

$$|\arg h(\sigma + iU)| \leq A(\log T),$$

for $\sigma \in (0, 1+\epsilon)$ and $h(\sigma + iU) \neq 0$.

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for $\sigma \in (0, 1+\epsilon)$ and $h(\sigma + iU) \neq 0$.
This gives the following bound for the difference of the second and third integral:

$$2A(\mu - \sigma') \log T.$$

Future works

Problem 1. Kadiri, Lumely and Ng used Jensen's formula and Backlund's trick to bound $|\arg|$. We propose to instead use them to directly bound the integral of the $|\arg|$.

Problem 2. Improve the constant and potentially change the shape of the upper bound.

Finally, after these steps, we are able to compile our bounds to obtain an upper bound for the number of zeros, $N(\sigma, T_1, T_2)$, in a given region.


THANK YOU!

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