Vertex operator algebras on modular curves

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Motivating question

VOAs are important in math and physics. They describe vector bundles on algebraic curves, and in fact on moduli spaces of curves.

Question

Do VOAs define bundles on p-adic curves?

Example

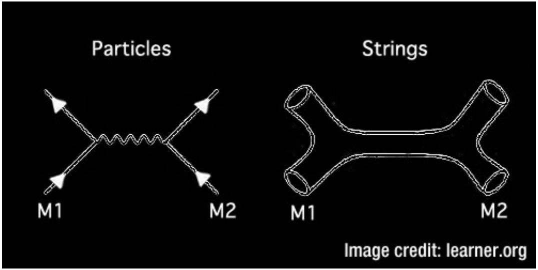
We have in mind, say, eigencurves, or limits of modular curves

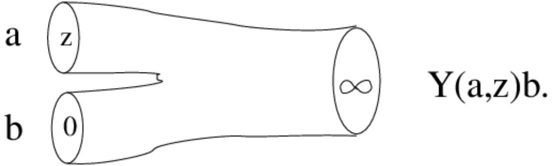
$$X_0(1) \leftarrow X_0(p) \leftarrow X_0(p^2) \leftarrow X_0(p^3) \leftarrow \cdots$$

In this talk we will explain the story on $X_0(1)$.

Vertex operator algebras on modular curves

Vertex operator algebras (VOAs) are vector spaces with countably many bilinear operations satisfying intricate identities. They first arose in physics of string theory and conformal field theory.





Vertex operators first arose in math in the study of infinite dimensional Lie algebras.

Example

Let $S = \mathbf{C}[y_{\frac{1}{2}}, y_{\frac{3}{2}}, y_{\frac{5}{2}}, \ldots]$ and define

$$Y = \exp\left(\sum_{n=\frac{1}{2},\frac{3}{2},\ldots} \frac{y_n}{n} x^n\right) \exp\left(-2 \sum_{n=\frac{1}{2},\frac{3}{2},\ldots} \frac{\partial}{\partial y_n} x^{-n}\right)$$

Set $Y = \sum_{j \in \frac{1}{2}\mathbf{Z}} Y_j x^j$, so each Y_j acts on S (not obvious).

Theorem (Lepowski-Wilson, 1978)

The operators 1, y_n, $\frac{\partial}{\partial y_n}$ *and* Y_j *span a Lie algebra in* $\mathrm{End}(S)$ *isomorphic with the affine Lie algebra* $\widehat{\mathfrak{sl}_2(\mathbf{C})}$.

Borcherds formalized properties of vertex operators and gave birth to mathematical formalism of VOAs. One goal was to explain:

Problem

Moonshine phenomenon: if j is the modular invariant

$$j(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

and if $r_1 < r_2 < r_3 < \cdots$ are the dimensions of the irreps of the monster simple group, then also

$$j(q) = \frac{r_1}{q} + 744 + (r_1 + r_2)q + (r_1 + r_2 + r_3)q^2 + (2r_2 + 2r_2 + r_3 + r_4)q^3 + \cdots$$

Why?

Definition

A VOA consists of a vector space $V = \bigoplus_{n \in \mathbf{Z}} V_n$ such that each V_n is finite dim and $V_n = 0$ if $n \ll 0$, equipped with

$$Y(\bullet, X) \colon V \to \mathrm{End}(V)[\![X, X^{-1}]\!], \qquad \underline{1} \in V, \qquad \omega \in V_2,$$

such that, if $Y(u, X) = \sum_{n \in \mathbf{Z}} u_n X^{-n-1}$, then

1. for all $u, v \in V$, there exists N with $u_n v = 0$ for all $n \geq N$;
2. $Y(\underline{1}, X) = 1$;
3. $Y(u, X)\underline{1} = u + O(X)$;
4. the coefficients of $Y(\omega, X)$ generate a representation of the Virasoro Lie algebra (more on next slide);
5. for all $a, b, c \in V$, and for all $m, n, l \in \mathbf{Z}$, the Jacobi identity holds:

$$\sum_{i=0}^{\infty} \binom{m}{i} (a_{l+i} b)_{m+n-i} = \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} \left\{ a_{m+l-i} b_{n+i} - b_{n+l-i} a_{m+i} \right\}.$$

Virasoro algebra action

Definition

The Virasoro Lie algebra $\mathcal{L}$ has basis $\{L(n) \mid n \in \mathbf{Z}\} \cup \{c\}$ with c central and

$$[L(m), L(n)] = (m-n)L(m+n) + \tfrac{m^3-m}{12}\delta_{m,-n}c.$$

Condition (5) above says that if $Y(\omega, X) = \sum_{n\in\mathbf{Z}} L(n)x^{-n-2}$,

1. $\mathcal{L}$ acts on V via the $L(n)$ and $c \mapsto c_V \in \mathbf{C}$ (the central charge of V);
2. $L(0)v = nv$ for $v \in V_n$;
3. $Y(L(-1)v, X) = \frac{d}{dX}Y(v, X)$.

Example

Notice that $\mathbf{C}L(0) \oplus \mathbf{C}L(1) \oplus \mathbf{C}L(-1) \cong \mathfrak{sl}_2(\mathbf{C})$. Indeed, if $\omega = \frac{1+\sqrt{-3}}{2}$ then

$$L(0) \mapsto \frac{1}{2}\begin{pmatrix} -\sqrt{-3} & 2 \\ 2 & \sqrt{-3} \end{pmatrix},$$

$$L(1) \mapsto \begin{pmatrix} -1 & -\omega \\ \bar{\omega} & 1 \end{pmatrix}, \qquad L(-1) \mapsto \begin{pmatrix} 1 & -\bar{\omega} \\ \omega & -1 \end{pmatrix}.$$

Jacobi identity

Jacobi identity (under suitable hypotheses) is equivalent to either

Proposition

For all $F(Z,W) \in \mathbf{C}[Z, W, Z^{-1}, W^{-1}, (Z-W)^{-1}]$ *one has in* $\mathbf{C}(\!(W)\!)$,

$$\begin{aligned} &\mathrm{Res}_{Z-W}(Y(Y(a,Z-W)b,W)\iota_{W,Z-W}(F(Z,W))) \\ =&\mathrm{Res}_Z(Y(a,Z)Y(b,W)\iota_{Z,W}(F(Z,W))) \\ &-\mathrm{Res}_Z(Y(b,W)Y(a,Z)\iota_{W,Z}(F(Z,W))) \end{aligned}$$

Proposition

Fields are mutually local: for all $a, b \in V$, *there exists* $N \geq 0$ *with*

$$(X-Z)^N[Y(a,X),Y(b,Z)] = 0.$$

Sources of VOAs

- commutative algebras with a derivation (simplest examples)
- Representations of infinite dimensional Lie algebras
- even unimodular lattices
- coset and orbifold constructions
- sub-VOAs e.g. commutators of well-known sub-VOAs
- Chiral de Rham complexes associated to smooth complex algebraic manifolds

Example: commutative rings

Definition

A VOA V is commutative if $Y(a,z)$ and $Y(b,w)$ commute as formal series in $\mathrm{End}(V)[[z,z^{-1},w,w^{-1}]]$ for all $a,b \in V$.

Proposition

A VOA V is commutative if and only if the image of the state-field correspondence lies in $\mathrm{End}(V)[[z]]$.

A commutative VOA is the same as a commutative ring V plus a derivation T:

- Given $a \in V$, ring product is $a \circ b := a(-1)b$ where $a(-1)$ is constant term of $Y(a,z)$.
- Derivation $T\colon V \to V$ determined by

$$Y(a,z) = \sum_{n \geq 0} T^n(a)\frac{z^n}{n!} = e^{zT}a.$$

$\therefore$ all interesting examples have poles/noncommutativity.

Example: Heisenberg algebra

Definition

The Heisenberg VOA is the graded vector-space

$$S = \mathbf{Q}[h_{-1}, h_{-2}, \ldots]$$

$\deg(h_{-n}) := n$. For $n > 0$ set $h_n = n\frac{d}{dh_{-n}}$. (Rep of Heisenberg Lie algebra.)

Remark

If S_n is degree n graded piece,

$$\dim(S_n) = p(n) = \text{number of partitions of } n \approx \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

The VOA structure of S is generated in a suitable sense by

- $Y(h_{-1}, z) := \sum_{n \in \mathbf{Z}} h_n z^{-n-1}$.
- Other fields $Y(h_{-n}, z)$ are derivatives of $Y(h_{-1}, z)$.
- Terms like h_{-1}^2 map to $Y(h_{-1}, z)^2$ except a normal ordering of the product must be used.

Number theory and VOAs

Characters of representations of many VOAs are modular forms.

Example

The monster module $V^\natural = \bigoplus_{n\geq 0} V_n^\natural$ with central charge $c = 24$, first constructed rigorously by Frenkel-Lepowsky-Meurmann, has a *unique* rep (itself) and its *character*

$$\chi(q) = q^{-c/24} \sum_{n=0}^{\infty} \left(\dim_{\mathbb{C}} V_n^\natural\right) q^n$$

is related to the j-invariant,

$$\chi = j - 744 = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

$V^\natural$ has an action of the monster, which explains moonshine.

This story generalizes to a nice class of VOAs. It includes various modularity results for reps of affine Lie algebras.

Strongly regular VOAs have:

1. finitely many simple modules and
2. semisimple rep category.

Definition

If V is strongly regular, M a simple module, define the character

$$\chi_M(q) := q^{h-c/24} \sum_{n \geq 0} (\dim_{\mathbf{C}} M_{n+h})\, q^n.$$

Here $c, h \in \mathbf{Q}$ are central charge of V and conformal weight of M.

Theorem (Zhu, 1996; Dong-Li-Mason, 2000.)

Let V be strongly regular, and $M_1, \ldots, M_m$ its simple modules. Then the character vector

$$\chi(q) = \begin{pmatrix} \chi_{M_1}(q) \\ \vdots \\ \chi_{M_m}(q) \end{pmatrix}$$

is a vector-valued modular form for an m-dim rep $\rho\colon \mathrm{PSL}_2(\mathbf{Z}) \to \mathrm{GL}_m(\mathbf{C})$.

Example

Let L be even self-dual unimodular lattice of rank d. For the associated lattice theory VOA V_L, built from the Heisenberg algebra and $\mathbf{C}[L]$, have

$$\chi_{V_L} = \frac{\theta_L}{\eta^d}$$

where θ_L is the *theta function* of L, a congruence form of weight $d/2$.

Example: Heisenberg algebra again

k	A000098	$\dim M_k(S)$
0	1	1
2	2	1
4	5	3
6	10	5
8	19	9
10	33	14
12	57	24
14	92	35
16	147	55
18	227	80
20	345	118

Table: Dimensions for Heisenberg-valued modular forms

- Early in project, we had a notion of V-valued modular forms.
- We computed the spaces $M_k(S)$ for small k.
- Noticed dimension counts are second difference of a well-known sequence
- A000098 of OEIS: partitions of n for which the parts 1, 2 and 3 have two colours.
- We wanted to understand $M(S)$ and explain this.
- To do so, we will have to introduce quasi-modular forms.

Vertex operator algebras on modular curves

Modular curves

Notation:

$$\Gamma := \mathrm{SL}_2(\mathbf{Z}),$$
$$\mathcal{H} := \{\tau = x + iy \in \mathbf{C} \mid y > 0\},$$
$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \tau := \frac{a\tau + b}{c\tau + d}, \qquad \tau \in \mathcal{H}, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

Definition

The modular curve of level one is $Y(1) := \Gamma \backslash \mathcal{H}$.

We will see that VOAs define bundles on curves.

Goals

Describe the bundles on $Y(1)$ associated to VOAs.

Bundles on modular curves

A bundle on $Y(1)$ is a trivial bundle on $\mathcal{H}$ with a compatible action of Γ: set $\mathcal{V} = \mathcal{H} \times V$ and assume given an action

$$\gamma \cdot (\tau, v) = (\gamma \cdot \tau, \kappa(\gamma, \tau)v)$$

for some $\kappa(\gamma, \tau) \in \mathrm{End}(V)$.

Lemma

The map κ satisfies the cocycle identity:

$$\kappa(\alpha\beta, \tau) = \kappa(\alpha, \beta \cdot \tau)\kappa(\beta, \tau).$$

Example

Modular forms arise from $j(\gamma, \tau) = c\tau + d$, where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.
Write $\mathcal{O}(k)$ for the bundle corresponding to $(V, \kappa) = (\mathbf{C}, j^k)$.

More cocycles

Example

Let $\rho\colon \Gamma \to \mathrm{GL}_n(\mathbf{C})$ be a representation. Then vector-valued modular forms associated to ρ are sections of the bundles

$$(V, \kappa) = (\mathbf{C}^n, j^k \rho).$$

Example

If V is a VOA, the Virasoro structure defines a cocycle

$$K(\gamma, \tau) := e^{-cj(\gamma,\tau)L(1)} j(\gamma, \tau)^{-2L(0)}.$$

The VOA bundle on $Y(1)$ corresponds to the pair (V, K).
We will explain where this comes from below.

Modular forms

The sections of a bundle (V, κ) are holomorphic functions

$$f\colon \mathcal{H} \to V$$

satisfying

$$f(\gamma\tau) = \kappa(\gamma, \tau) f(\tau).$$

Call this a V-valued modular form, leaving κ implicit. A V-valued modular form of weight $k \in \mathbf{Z}$ is a section of $(V, j^k \kappa)$.

Definition

$M_k(V) = M_k(V, \kappa)$ is the space of modular forms associated to (V, κ) of weight k, and

$$M(V) = M(V, \kappa) := \bigoplus_{k \in \mathbf{Z}} M_k(V, \kappa).$$

Consider the case $(V, \kappa) = (\mathbf{C}, j)$.

Example

Ring of modular forms of level one is $M = \mathbf{Q}[E_4, E_6]$ where

$$E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n \geq 1} \sigma_{2k-1}(n) q^n, \qquad q = e^{2\pi i \tau},$$

$$\sigma_k(n) = \sum_{d|n} d^k, \qquad \frac{t}{e^t - 1} = \sum_{k \geq 0} \frac{B_k}{k!} t^k.$$

Example

E_2 is only quasi-modular, satisfying:

$$E_2(\gamma\tau) = j(\gamma, \tau)^2 E_2(\tau) + \frac{12cj(\gamma, \tau)}{2\pi i}.$$

The ring of quasi-modular forms of level one is $Q = \mathbf{C}[E_2, E_4, E_6]$.

Vertex operator algebras on modular curves

Virasoro action

VOAs allow one to define bundles on smooth curves by exponentiating the Virasoro action:

Definition

Define $\mathcal{O} := \mathbf{C}[\![Z]\!]$ and recall that

$$\operatorname{Aut}\mathcal{O} = \{a_1 Z + a_2 Z^2 + a_3 Z^3 + \cdots \in \mathcal{O} \mid a_1 \neq 0\}.$$

Lemma (Lemma 5.2.2 of Frenkel-Ben Zvi)

If V is a VOA, there is an action of $\operatorname{Aut}\mathcal{O}$ on V defined by

$$f(z) = \sum_{n\geq 1} a_n Z^N = \exp\left(\sum_{i>0} v_i Z^{i+1}\frac{d}{dz}\right) v_0 Z,$$

$$R(f) = \exp\left(-\sum_{i>0} v_j L_j\right) v_0^{-L_0}.$$

The bundle construction

Definition

If X is a smooth complex curve, let $\mathcal{A}\mathrm{ut}_X$ denote its torsor of (formal) local coordinates.

Example

If $X = \mathbf{C}$ then for all $x \in X$ we have

$$\mathcal{A}\mathrm{ut}_{X,x} = \mathrm{Aut}\,\mathbf{C}[\![Z - x]\!] \cong \mathrm{Aut}\,\mathcal{O}.$$

Definition

The VOA bundle associated to a VOA V on X is

$$\mathcal{V}_X := \mathcal{A}\mathrm{ut}_X \times_{\mathrm{Aut}\,\mathcal{O}} V.$$

The case of the modular curve

We want to understand the bundle

$$\mathcal{V}_{Y(1)} = \mathcal{A}ut_{Y(1)} \times_{\mathrm{Aut}\,\mathcal{O}} V \cong \Gamma\backslash\left((\mathrm{Aut}\,\mathcal{O} \times \mathcal{H}) \times_{\mathrm{Aut}\,\mathcal{O}} V\right) \cong \Gamma\backslash(\mathcal{H} \times V).$$

It is described by a cocycle. Unwinding the action yields:

Theorem (BCFMN, 2025)

The bundle $\mathcal{V}_{Y(1)}$ corresponds to the cocycle

$$K(\gamma, \tau) = e^{-cj(\gamma,\tau)L(1)} j(\gamma, \tau)^{-2L(0)}.$$

VOA valued modular forms

A V-valued modular form of weight k is a function $f\colon \mathcal{H} \to V$ satisfying the transformation law

$$f(\gamma\tau) = j(\gamma,\tau)^k K(\gamma,\tau) f(\tau).$$

Let $M(V)$ be the space of all such forms. It is an M-module.

Theorem (BCFMN, 2025)

There is a graded isomorphism of M-modules

$$P := \exp\left(-\frac{2\pi i}{12} E_2 L(1)\right) : M \otimes V^{(2)} \to M(V).$$

Therefore,

$$\mathcal{V}_{Y(1)} \cong \bigoplus_{k\geq 0} \bigoplus_{\ell=0}^{k} (k-\ell)\mathcal{O}(2\ell)$$

Modes and quasi-modular forms

If $f, g \in M(V)$, then they have modes $f_n g$ coming from V. But $M(V)$ is not closed under modes!

Theorem (BCFMN, 2025)

For any $f, g \in M(V)$, the modes $f_n g$ are V-valued quasimodular forms. If $Q(V)$ is the module of such objects, there is an isomorphism of $Q = \mathbf{C}[E_2, E_4, E_6]$-modules

$$Q \otimes V^{(2)} \cong Q(V).$$

In fact, $Q(V)$ has a natural structure of quasiVOA.

Moral

Quasi-modularity arises by taking modes of VOA-valued modular forms.

Example: Heisenberg algebra yet again

k	$\dim Q_k(S) = \text{A000098}$	$\dim M_k(S)$
0	1	1
2	2	1
4	5	3
6	10	5
8	19	9
10	33	14
12	57	24
14	92	35
16	147	55
18	227	80
20	345	118

Table: Dimensions for Heisenberg-valued modular forms

- Now we understand our computations are a reflection of

$$M(S) \cong M \otimes S \subseteq Q(S) \cong Q \otimes S.$$

- $\therefore$ A000098 counts dimensions of Heisenberg algebra valued quasi-modular forms.
- Recall A000098 counts parititons of n where 1, 2 and 3 have two colours.
- The colours denote if they come from S ($h(-1)$, $h(-2)$, $h(-3)$) or Q (E_2, E_4, E_6).

The map P^{-1}

We can use the isomorphism $P^{-1} = \exp(\frac{2\pi i}{12} E_2 L(1))$ to find explicit bases:

k	Basis Vectors for $M_k(S)$
0	1
2	$h(-1)$
4	$E_4, \quad h(-1)^2, \quad E_2 h(-1) - \frac{3}{\pi i} h(-2)$
6	$E_6, \quad E_4 h(-1), \quad h(-1)^3,$ $E_2^2 h(-1) - \frac{6}{\pi i} E_2 h(-2) - \frac{12}{\pi^2} h(-3),$ $E_2 h(-1)^2 - \frac{3}{\pi i} h(-2) h(-1)$
8	$E_4^2, \quad E_6 h(-1), \quad E_4 h(-1)^2, \quad h(-1)^4, E_4 h(-2) - \frac{\pi i}{3} E_4 E_2 h(-1),$ $h(-4) + \frac{\pi^3 i}{54} E_2^3 h(-1) - \frac{\pi}{6} E_2^2 h(-2) - \frac{2\pi i}{3} E_2 h(-3),$ $h(-3) h(-1) - \frac{\pi^2}{12} E_2^2 h(-1)^2 - \frac{\pi i}{2} E_2 h(-2) h(-1),$ $h(-2)^2 - \frac{\pi^2}{9} E_2^2 h(-1)^2 - \frac{2\pi i}{3} E_2 h(-2) h(-1),$ $h(-2) h(-1)^2 - \frac{\pi i}{3} E_2 h(-1)^3$

Future directions

Next steps in this project:

- unravel representation theory of VOAs, Zhu's theorem, et cetera, in a similar way.
- lift these constructions up the modular tower

$$\underbrace{X_0(1)}_{\text{We are here!}} \leftarrow X_0(p) \leftarrow X_0(p^2) \leftarrow X_0(p^3) \leftarrow \cdots$$

- use the result to study the p-adic properties of VOAs.

Question

The character map on S surjects onto $\eta^{-1}\mathbf{C}[E_2, E_4, E_6]$. Does the p-adic version surject onto the ring of p-adic modular forms?

Thanks for listening!