Moments of the Hurwitz zeta function on the critical line

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(partly joint with Winston Heap and Trevor Wooley)

Overview of the Talk

1. What is the Hurwitz zeta function?
2. Moments of the Hurwitz zeta function for rational shifts
3. Moments of the Hurwitz zeta function for irrational shifts

The zeta functions of Hurwitz and Riemann

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$$\zeta(s, \alpha) = \sum_{n \geqslant 0} \frac{1}{(n + \alpha)^s},$$

for $\sigma > 1$. This is the shifted integer analogue for the (usual) zeta function of Riemann, $\zeta(s) = \zeta(s, 1)$, given by

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s},$$

for $\sigma > 1$.

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- They both satisfy a "functional equation".

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These can both be viewed as manifestations of the Poisson summation formula.

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- The same is true in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$ for rational shifts (Voronin, 1976) and transcendental shifts (Gonek, 1979). This is open for algebraic irrationals!

Moments of $\zeta(s)$

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are called the moments of $\zeta(s)$. Estimates for $M_k(T)$ are useful in several problems in analytic number theory; in particular,

$$\zeta(\tfrac{1}{2} + it) \ll_\epsilon |t|^\epsilon \iff M_k(T) \ll_{k,\epsilon} T^{1+\epsilon},$$

where the left hand side here is the Lindelöf hypothesis.

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- (Keating–Snaith, 2000) gave a conjecure for c_k for every $k > 0$ by the analogy with random matrix theory.

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We will justify this expectation for rational α.

Things are more complicated for irrational α – if time permits, we will return to this later.

What is known about $M_k(T;\alpha)$?

The classical mean-square methods for $\zeta(s)$ apply also to $\zeta(s,\alpha)$. (Rane, 1980) showed that uniformly for all $0 < \alpha \leqslant 1$,

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for an explicit constant $B(\alpha)$. The error term has been improved a few times; the best error is due to (Zhan, 1993).

The uniformity in α here is perhaps a coincidence – more on this later.

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Conjecture (S., 2021+)

Let $k \geqslant 0$ and $\alpha = a/q$ be as above. Then,

$$\int_T^{2T} \left|\zeta\left(\tfrac{1}{2} + it, \alpha\right)\right|^{2k} dt \sim c_k(\alpha) T (\log T)^{k^2},$$

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Note that $c_k(\alpha)$ does not depend on a!

Reduction to mean-square of $\mathcal{L}^{\ell}(s)$

By orthogonality of Dirichlet characters, we have for $\alpha = a/q$ and $\sigma > 1$,

$$\zeta(s,\alpha) = \sum_{n \geqslant 0} \frac{1}{(n+\alpha)^s} = \sum_{n \geqslant 0} \frac{q^s}{(qn+a)^s}$$

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&= q^s \sum_{m \geqslant 1} \frac{1}{m^s} \left(\frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a)\chi(m) \right) \\
&= \frac{q^s}{\varphi(q)} \sum_{\chi} \overline{\chi}(a) L(s,\chi).
\end{aligned}$$

By analytic continuation, this holds everywhere in $\mathbb{C}$.

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Thus, by the multinomial theorem,

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where $\binom{k}{\boldsymbol{\ell}} = \frac{k!}{\prod_\chi \ell_\chi!}$ are the multinomial coefficients.

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Using $|\zeta(s,\alpha)|^{2k} = \zeta(s,\alpha)^k \overline{\zeta(s,\alpha)}^k$,

$$|\zeta(s,\alpha)|^{2k} = \frac{q^{2k\sigma}}{\varphi(q)^{2k}} \sum_{\substack{|\boldsymbol{\ell}^{(1)}|=k \\ |\boldsymbol{\ell}^{(2)}|=k}} \binom{k}{\boldsymbol{\ell}^{(1)}} \binom{k}{\boldsymbol{\ell}^{(2)}} \mathfrak{s}(a; \boldsymbol{\ell}^{(1)}, \boldsymbol{\ell}^{(2)}) \mathcal{L}^{\boldsymbol{\ell}^{(1)}}(s) \overline{\mathcal{L}^{\boldsymbol{\ell}^{(2)}}(s)},$$

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where $\mathfrak{s}(a;\boldsymbol{\ell}^{(1)},\boldsymbol{\ell}^{(2)})$ is a complex number of magnitude 1. If we now put $s = \frac{1}{2} + it$ and integrate over $t \in [T, 2T]$, we get $M_k(T;\alpha)$ on the left, while on the right we expect terms with $\boldsymbol{\ell}^{(1)} \neq \boldsymbol{\ell}^{(2)}$ to oscillate very fast.

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Reduction to mean-square of $\mathcal{L}^{\boldsymbol{\ell}}(s)$

This gives, heuristically,

$$M_k(T;\alpha) \approx \frac{q^k}{\varphi(q)^{2k}} \sum_{|\boldsymbol{\ell}|=k} \binom{k}{\boldsymbol{\ell}}^2 \int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}}\left(\tfrac{1}{2}+it\right)\right|^2 dt.$$

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Note that the right hand side does not depend on a, as predicted in our conjecture!

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Since Dirichlet L-functions fall in both these classes, their results apply also to $\mathcal{L}^{\ell}(s)$ (and, in fact, also to the Dedekind zeta functions $\zeta_K(s)$ of a Galois number field K).

The main theorem

Theorem (S., 2021+)

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$$\frac{1}{T}\int_T^{2T} |\mathcal{L}^{\boldsymbol{\ell}}(\tfrac{1}{2}+it)|^2\,dt \sim_{q,k} c_{\boldsymbol{\ell}}(q)\left\{\prod_{\chi} (\log q^*(\chi)T)^{\ell_\chi^2}\right\},$$

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where $c_{\boldsymbol{\ell}}(q)$ *is given by*

$$\prod_p\left\{\left(1-\frac{1}{p}\right)^{\lambda}\sum_{m=0}^{\infty}\frac{|d_{\boldsymbol{\ell}}(p^m)|^2}{p^m}\right\}\prod_{\chi}\frac{G(\ell_\chi+1)^2}{G(2\ell_\chi+1)}.$$

Here $\lambda=\sum_\chi \ell_\chi^2$, $G(\cdot)$ *is the Barnes* G*-function and* $q^*(\chi)$ *is the conductor of* $L(s,\chi)$.

The main theorem

Theorem (S., 2021+)

Under some reasonable conjectures[a] *we have that for any* ℓ*,*

$$\frac{1}{T}\int_T^{2T} |\mathcal{L}^{\boldsymbol{\ell}}(\tfrac{1}{2}+it)|^2 \, dt \sim_{q,k} c_{\boldsymbol{\ell}}(q)\left\{ \prod_{\chi} (\log q^*(\chi) T)^{\ell_\chi^2} \right\},$$

where $c_{\boldsymbol{\ell}}(q)$ *is given by*

$$\prod_p \left\{ \left(1-\frac{1}{p}\right)^{\lambda} \sum_{m=0}^{\infty} \frac{|d_{\boldsymbol{\ell}}(p^m)|^2}{p^m} \right\} \prod_{\chi} \frac{G(\ell_\chi+1)^2}{G(2\ell_\chi+1)}.$$

Here $\lambda = \sum_\chi \ell_\chi^2$*,* $G(\cdot)$ *is the Barnes* G*-function and* $q^*(\chi)$ *is the conductor of* $L(s,\chi)$*.*

[a] To be described; based on the approach of (Gonek–Hughes–Keating, 2007) instead of the CFKRS recipe.

Theorem $\implies$ Conjecture

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$$M_k(T;\alpha) \approx \frac{q^k}{\varphi(q)^{2k}} \sum_{|\boldsymbol{\ell}|=k} \binom{k}{\boldsymbol{\ell}}^2 \int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}}\left(\tfrac{1}{2}+it\right)\right|^2 dt$$

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Thus, our main conjecture follows from this theorem after some book-keeping.

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$$\mathcal{L}^{\ell}(s) \approx \mathcal{P}_X^{\ell}(s)\mathcal{Z}_X^{\ell}(s),$$

where

$\mathcal{P}_X^{\ell}(s)$	approximate Euler product	Primes	$p \leqslant X$
$\mathcal{Z}_X^{\ell}(s)$	approximate Hadamard product	Zeroes	$\lvert\rho - t\rvert \leqslant \frac{1}{\log X}$

The splitting conjecture

One expects $\mathcal{P}_X^{\ell}(s)$ and $\mathcal{Z}_X^{\ell}(s)$ to behave like independent random variables for $X = o(T)$.

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Conjecture (Splitting)

Let $X, T \to \infty$ with $X \ll_\epsilon (\log T)^{2-\epsilon}$. Then, for any tuple of nonnegative integers ℓ indexed by characters modulo q, we have for $s = 1/2 + it$,

$$\frac{1}{T}\int_T^{2T} \left|\mathcal{L}^{\ell}(s)\right|^2 dt \sim \left(\frac{1}{T}\int_T^{2T} \left|\mathcal{P}_X^{\ell}(s)\right|^2 dt\right) \times \left(\frac{1}{T}\int_T^{2T} \left|\mathcal{Z}_X^{\ell}(s)\right|^2 dt\right).$$

Mean-square of $\mathcal{P}_X^{\ell}(s)$

Theorem (S., 2021+)

For integer $\ell_\chi \geqslant 0$ *such that* $|\boldsymbol{\ell}| = \sum_\chi \ell_\chi = k$*, further, suppose that* $2 \leqslant X \ll_\epsilon (\log T)^{2-\epsilon}$.

$$\frac{1}{T}\int_T^{2T} |\mathcal{P}_X^{\boldsymbol{\ell}}(\tfrac{1}{2}+it)|^2\,dt = b(\boldsymbol{\ell})F_X(\boldsymbol{\ell})\left(1+\mathcal{O}_{q,k,\epsilon}\left(\frac{1}{\log X}\right)\right)$$

where $b(\boldsymbol{\ell})$ *is an explicit Euler product independent of* X*, and*

$$F_X(\boldsymbol{\ell}) = (e^\gamma \log X)^\lambda \prod_p \left(1-\frac{1}{p}\right)^{\lambda-|d_{\boldsymbol{\ell}}(p)|^2}.$$

Here γ *is the Euler-Mascheroni constant,* $d_{\boldsymbol{\ell}}(n)$ *is the coefficient of* n^{-s} *in the Dirichlet series for* $\mathcal{L}^{\boldsymbol{\ell}}(s)$*, and* $\lambda = \sum_\chi \ell_\chi^2$.

Mean-square of $\mathcal{Z}_X^{\boldsymbol{\ell}}(s)$

The random matrix theory analogy gives us the following conjecture

Conjecture

Suppose that $X, T \to \infty$ with $X \ll_\epsilon (\log T)^{2-\epsilon}$. Then, for $\boldsymbol{\ell}$ as before,

$$\frac{1}{T}\int_T^{2T} |\mathcal{Z}_X^{\boldsymbol{\ell}}(\tfrac{1}{2}+it)|^2\, dt \sim \prod_\chi \left[\frac{G(\ell_\chi+1)^2}{G(2\ell_\chi+1)} \left(\frac{\log q^*(\chi) T}{e^\gamma \log X} \right)^{\ell_\chi^2} \right],$$

where $G(\cdot)$ is the Barnes G-function, and $q^*(\chi)$ is the conductor of χ.

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Here $L(s,\chi)$ forms a unitary family in the t-aspect, and so we model each $Z(s,\chi)$ in $\mathcal{Z}_X^{\ell}(s)$ by unitary matrices chosen independently and uniformly with respect to the Haar measure.

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Here $L(s,\chi)$ forms a unitary family in the t-aspect, and so we model each $Z(s,\chi)$ in $\mathcal{Z}_X^{\ell}(s)$ by unitary matrices chosen independently and uniformly with respect to the Haar measure. The size of the unitary matrices are chosen appropriately so that it matches the approximate mean density of the zeroes.

Other evidence for the conjecture

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1. We show that $T(\log T)^{k^2} \ll_{k,\alpha} M_k(T;\alpha) \ll_{k,\alpha,\epsilon} T(\log T)^{k^2+\epsilon}$. The upper bound is conditional on GRH for all Dirichlet L-functions mod q, while the lower bound is unconditional.
2. We prove some small ($|\boldsymbol{\ell}| \leqslant 2$) cases of the splitting and random matrix theory conjectures using standard techniques.
3. We verify that our conjectural constants match up in all the cases where asymptotics are known.

Conjectures for irrational shifts

In ongoing work, we have made the following conjecture.

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Conjecture (Heap–S., 2022++)

Let $k \in \mathbb{N}$ and $0 < \alpha \leqslant 1$ be an irrational number. Then for algebraic α of degree $d \geqslant k$ and almost all transcendental α we have

$$M_k(T;\alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^{2k} dt \sim k!\, T(\log T)^k$$

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Note that this conjecture suggests Gaussian behaviour!

Pseudomoments of $\zeta(s, \alpha)$, $\alpha \notin \mathbb{Q}$

To gain some insight into this conjecture, we first consider the case that α is transcendental and consider the pseudomoments $M_k'(N;\alpha)$ defined by

$$M_k'(N;\alpha) = \lim_{T\to\infty} \frac{1}{T}\int_T^{2T} \left| \sum_{0\leqslant n\leqslant N} \frac{1}{(n+\alpha)^{\frac{1}{2}+it}} \right|^{2k} dt$$

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$$= \sum_{\substack{0\leqslant n_1,\cdots,n_k,m_1,\cdots,m_k\leqslant N \\ (n_1+\alpha)\cdots(n_k+\alpha)=(m_1+\alpha)\cdots(m_k+\alpha)}} \frac{1}{(n_1+\alpha)\cdots(n_k+\alpha)}.$$

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$$M_k'(N;\alpha) \sim k! \sum_{\substack{0 \leqslant n_j \leqslant N \\ 1 \leqslant j \leqslant k}} \frac{1}{(n_1+\alpha)\cdots(n_k+\alpha)}$$

$$= k! \Bigg(\sum_{0 \leqslant n \leqslant N} \frac{1}{n+\alpha} \Bigg)^k \sim k!(\log N)^k.$$

The fourth moment of $\zeta(s, \alpha)$ for irrational α

Theorem (Heap–S, 2022++)

Let $0 < \alpha < 1$ be an irrational number. Then, under certain Diophantine conditions[a], we have that

$$M_2(T;\alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^4 dt$$

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as $T \to \infty$.

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In particular, our Diophantine conditions appear to be satisfied by almost all α, so this verifies the previous conjecture for $k = 2$.

Why do we need a Diophantine criterion?

After applying an approximate functional equation, and focusing just on the first piece, which terms contribute depend on our control over the harmonics

$$\int_T^{2T} \left[\frac{(n_1 + \alpha)(n_2 + \alpha)}{(n_3 + \alpha)(n_4 + \alpha)} \right]^{it} dt.$$

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It's hard to rule out a main contribution arising from an off-diagonal term $\{n_1, n_2\} \neq \{n_3, n_4\}$ with

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or, in other words, from terms with

$$\alpha \approx \frac{n_1 n_2 - n_3 n_4}{n_1 + n_2 - n_3 - n_4}.$$

Our Diophantine assumptions let us show that this does not happen frequently enough to give a main term.

Pseudomoments for algebraic irrational shift parameters

Now, suppose that α is algebraic of degree $d \geqslant 2$. If $k \leqslant d$, then the argument for transcendental α goes through to give

$$M'_k(N;\alpha) \sim k!(\log N)^k.$$

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For $k > d$, however, we can now have solutions to

$$\prod_{j=1}^{k}(n_j+\alpha) = \prod_{j=1}^{k}(m_j+\alpha).$$

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We thus need to understand a fairly complicated problem of a logarithmic weight count of integer points in a variety. It is still unclear how these weighted counts behave for large k, but something can be said if we drop the logarithmic weights $1/(n_j+\alpha)$.

The unweighted integer-point counting problem

Theorem (Heap–S.–Wooley, 2022; independently Bourgain–Garaev–Konyagin–Shparlinski, 2014)

Let $k \in \mathbb{N}$ and $\epsilon > 0$. Suppose that $\alpha \in \mathbb{C}$ is algebraic of degree d over $\mathbb{Q}$ where $k > d$. Then, one has

$$\sum_{\nu \in \mathbb{Z}[\alpha]} \tau_k(\nu; X, \alpha)^2 = \sum_{\substack{1 \leqslant n_1, \cdots, n_k, m_1, \cdots, m_k \leqslant N \\ (n_1+\alpha)\cdots(n_k+\alpha) = (m_1+\alpha)\cdots(m_k+\alpha)}} 1$$

$$= T_k(X) + O_{k,\alpha,\epsilon}(X^{k-d+1+\epsilon}).$$

Here $T_k(X) = k! X^k + O_k(X^{k-1})$ is the number of pairs $(\boldsymbol{n}, \boldsymbol{m})$ with $1 \leqslant n_j, m_j \leqslant X$, $1 \leqslant j \leqslant k$ and $\boldsymbol{n}$ is a permutation of $\boldsymbol{m}$.

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The error term here may be omitted if, instead, $k \leqslant d$, or if α is transcendental.

Thank You!