Projective planes and Hadamard matrices

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University of Lethbridge
Number Theory and Combinatorics Seminar

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A finite projective plane of order2

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- 7 points,

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- 7 points, 7 lines,

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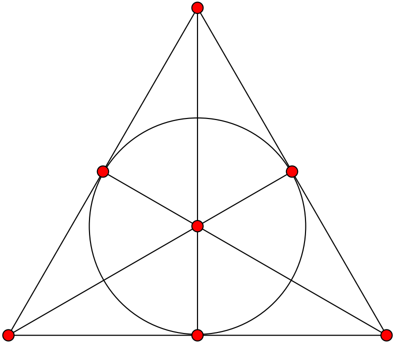

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$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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Existence of a projective plane of order 10

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Is a math proof a proof if no one can check it?

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Bush-type Hadamard matrices

Example: A Bush-type Hadamard matrix of order 16

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & - & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - & 1 & - & 1 & 1 & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 \\
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1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 \\
- & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - \\
- & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 \\
1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
- & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 \\
- & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 \\
- & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 \\
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If there is a projective plane of order 10, then there is a **symmetric** *Bush-type Hadamard matrix of order 100.*

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Balancedly Splittable Hadamard matrices

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$$H = \left[\begin{array}{c} H_1 \\ \hline H_2 \end{array}\right] = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \hline 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{array}\right]$$

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$$H_1^t H_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

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$$H_2^tH_2=\begin{bmatrix}3 & - & - & -\\- & 3 & - & -\\- & - & 3 & -\\- & - & - & 3\end{bmatrix}$$

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$$H = \left[\frac{H_1}{H_2}\right] = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \hline 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{array}\right]$$

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Every normalized Hadamard matrix is balancedly splittable in this way.

Here is a twin balancedly splitted Hadamard matrix of order 16:

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$$
\begin{bmatrix} H_0 \\ \hline H_1 \\ \hline H_2 \end{bmatrix} =
\left[\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - \\
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\hline
1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\
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1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 \\
1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 \\
1 & - & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 \\
1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 \\
\hline
1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 \\
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1 & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - \\
1 & - & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - \\
1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & -
\end{array}\right]
$$

$$H_0 = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&-&-&-&-&1&1&1&1&-&-&-&- \\ 1&1&1&1&1&1&1&1&-&-&-&-&-&-&-&- \\ 1&1&1&1&-&-&-&-&-&-&-&-&1&1&1&1 \end{bmatrix}$$

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$$H_1 = \begin{bmatrix} 1&-&1&-&1&-&1&-&1&-&1&-&1&-&1&- \\ 1&1&-&-&1&1&-&-&1&1&-&-&1&1&-&- \\ 1&-&-&1&1&-&-&1&1&-&-&1&1&-&-&1 \\ 1&1&-&-&-&-&1&1&1&1&-&-&-&-&1&1 \\ 1&-&1&-&1&-&1&-&-&1&-&1&-&1&-&1 \\ 1&-&-&1&-&1&1&-&-&1&1&-&1&-&-&1 \end{bmatrix}$$

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$$H_2 = \begin{bmatrix} 1&-&1&-&-&1&-&1&1&-&1&-&-&1&-&1 \\ 1&-&-&1&-&1&1&-&1&-&-&1&-&1&1&- \\ 1&1&-&-&1&1&-&-&-&-&1&1&-&-&1&1 \\ 1&-&-&1&1&-&-&1&-&1&1&-&-&1&1&- \\ 1&-&1&-&-&1&-&1&-&1&-&1&1&-&1&- \\ 1&1&-&-&-&-&1&1&-&-&1&1&1&1&-&- \end{bmatrix}$$

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1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 \\
1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 \\
1 & - & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 \\
1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 \\
\hline
1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 \\
1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - \\
1 & - & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - \\
1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & -
\end{array}\right]$$

$$H_1^t H_1 = \begin{bmatrix}
6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 \\
\bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
\bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
\bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} \\
\bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 \\
2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6
\end{bmatrix}$$

$$
H_1^t H_1 = \begin{bmatrix}
6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 \\
\bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
\bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
\bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} \\
\bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 \\
2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6
\end{bmatrix}
\quad
H_2^t H_2 = \begin{bmatrix}
6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
\bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 \\
\bar{2} & \bar{2} & \bar{2} & 6 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 \\
\bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
\bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 6 & 2 & \bar{2} & 2 & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 6
\end{bmatrix}
$$

$$H_1^t H_1 = \begin{bmatrix}
6&\bar{2}&\bar{2}&\bar{2}&2&\bar{2}&2&\bar{2}&2&2&\bar{2}&\bar{2}&2&\bar{2}&\bar{2}&2\\
\bar{2}&6&\bar{2}&\bar{2}&\bar{2}&2&\bar{2}&2&2&2&\bar{2}&\bar{2}&\bar{2}&2&2&\bar{2}\\
\bar{2}&\bar{2}&6&\bar{2}&2&\bar{2}&2&\bar{2}&\bar{2}&\bar{2}&2&2&\bar{2}&2&2&\bar{2}\\
\bar{2}&\bar{2}&\bar{2}&6&\bar{2}&2&\bar{2}&2&\bar{2}&\bar{2}&2&2&2&\bar{2}&\bar{2}&2\\
2&\bar{2}&2&\bar{2}&6&\bar{2}&\bar{2}&\bar{2}&2&\bar{2}&\bar{2}&2&2&2&\bar{2}&\bar{2}\\
\bar{2}&2&\bar{2}&2&\bar{2}&6&\bar{2}&\bar{2}&\bar{2}&2&2&\bar{2}&2&2&\bar{2}&\bar{2}\\
2&\bar{2}&2&\bar{2}&\bar{2}&\bar{2}&6&\bar{2}&\bar{2}&2&2&\bar{2}&\bar{2}&\bar{2}&2&2\\
\bar{2}&2&\bar{2}&2&\bar{2}&\bar{2}&\bar{2}&6&2&\bar{2}&\bar{2}&2&\bar{2}&\bar{2}&2&2\\
2&2&\bar{2}&\bar{2}&2&\bar{2}&\bar{2}&2&6&\bar{2}&\bar{2}&\bar{2}&2&\bar{2}&2&\bar{2}\\
2&2&\bar{2}&\bar{2}&\bar{2}&2&2&\bar{2}&\bar{2}&6&\bar{2}&\bar{2}&\bar{2}&2&\bar{2}&2\\
\bar{2}&\bar{2}&2&2&\bar{2}&2&2&\bar{2}&\bar{2}&\bar{2}&6&\bar{2}&2&\bar{2}&2&\bar{2}\\
\bar{2}&\bar{2}&2&2&2&\bar{2}&\bar{2}&2&\bar{2}&\bar{2}&\bar{2}&6&\bar{2}&2&\bar{2}&2\\
2&\bar{2}&\bar{2}&2&2&2&\bar{2}&\bar{2}&2&\bar{2}&2&\bar{2}&6&\bar{2}&\bar{2}&\bar{2}\\
\bar{2}&2&2&\bar{2}&2&2&\bar{2}&\bar{2}&\bar{2}&2&\bar{2}&2&\bar{2}&6&\bar{2}&\bar{2}\\
\bar{2}&2&2&\bar{2}&\bar{2}&\bar{2}&2&2&2&\bar{2}&2&2&2&\bar{2}&6&\bar{2}\\
2&\bar{2}&\bar{2}&2&\bar{2}&\bar{2}&2&2&\bar{2}&2&\bar{2}&2&\bar{2}&\bar{2}&\bar{2}&6
\end{bmatrix}
\quad
H_2^t H_2 = \begin{bmatrix}
6&\bar{2}&\bar{2}&\bar{2}&\bar{2}&2&\bar{2}&2&\bar{2}&\bar{2}&2&2&\bar{2}&2&2&\bar{2}\\
\bar{2}&6&\bar{2}&\bar{2}&2&\bar{2}&2&\bar{2}&\bar{2}&\bar{2}&2&2&2&\bar{2}&\bar{2}&2\\
\bar{2}&\bar{2}&6&\bar{2}&\bar{2}&2&\bar{2}&2&2&2&\bar{2}&\bar{2}&2&\bar{2}&\bar{2}&2\\
\bar{2}&\bar{2}&\bar{2}&6&2&\bar{2}&2&\bar{2}&2&2&\bar{2}&\bar{2}&\bar{2}&2&2&\bar{2}\\
\bar{2}&2&\bar{2}&2&6&\bar{2}&\bar{2}&\bar{2}&\bar{2}&2&2&\bar{2}&\bar{2}&\bar{2}&2&2\\
2&\bar{2}&2&\bar{2}&\bar{2}&6&\bar{2}&\bar{2}&2&\bar{2}&\bar{2}&2&\bar{2}&\bar{2}&2&2\\
\bar{2}&2&\bar{2}&2&\bar{2}&\bar{2}&6&\bar{2}&2&\bar{2}&\bar{2}&2&2&2&\bar{2}&\bar{2}\\
2&\bar{2}&2&\bar{2}&\bar{2}&\bar{2}&\bar{2}&6&\bar{2}&2&2&\bar{2}&2&2&\bar{2}&\bar{2}\\
\bar{2}&\bar{2}&2&2&\bar{2}&2&2&\bar{2}&6&\bar{2}&\bar{2}&\bar{2}&\bar{2}&2&\bar{2}&2\\
\bar{2}&\bar{2}&2&2&2&\bar{2}&\bar{2}&2&\bar{2}&6&\bar{2}&\bar{2}&2&\bar{2}&2&\bar{2}\\
2&2&\bar{2}&\bar{2}&2&\bar{2}&\bar{2}&2&\bar{2}&\bar{2}&6&\bar{2}&\bar{2}&2&\bar{2}&2\\
2&2&\bar{2}&\bar{2}&\bar{2}&2&2&\bar{2}&\bar{2}&\bar{2}&\bar{2}&6&2&\bar{2}&2&\bar{2}\\
\bar{2}&2&2&\bar{2}&\bar{2}&\bar{2}&2&2&\bar{2}&2&\bar{2}&2&6&\bar{2}&\bar{2}&\bar{2}\\
2&\bar{2}&\bar{2}&2&\bar{2}&\bar{2}&2&2&2&\bar{2}&2&\bar{2}&\bar{2}&6&\bar{2}&\bar{2}\\
2&\bar{2}&\bar{2}&2&2&2&\bar{2}&\bar{2}&\bar{2}&2&\bar{2}&2&\bar{2}&\bar{2}&6&\bar{2}\\
\bar{2}&2&2&\bar{2}&2&2&\bar{2}&\bar{2}&2&\bar{2}&2&\bar{2}&\bar{2}&\bar{2}&\bar{2}&6
\end{bmatrix}$$

The corresponding angle between lines is $arccos(\frac{1}{3})$ for both sets of lines.

$$H_1^t H_1 = \begin{bmatrix}
6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 \\
\bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
\bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
\bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} \\
\bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 \\
2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6
\end{bmatrix}
\quad
H_2^t H_2 = \begin{bmatrix}
6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
\bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 \\
\bar{2} & \bar{2} & \bar{2} & 6 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 \\
\bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
\bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 6 & 2 & \bar{2} & 2 & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 6
\end{bmatrix}$$

The corresponding angle between lines is $arccos(\frac{1}{3})$ for both sets of lines.

Sixteen Equiangular Lines in $\mathbb{R}^{10}$.

Sixteen Equiangular Lines in $\mathbb{R}^{10}$.

$$\begin{bmatrix} H_0 \\ \hline H_1 \end{bmatrix} = \left[\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
\hline
1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 \\
1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 \\
1 & - & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 \\
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\end{array}\right]$$

$$H_0^t H_0 + H_1^t H_1 = \begin{bmatrix}
10 & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 \\
2 & 10 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} \\
2 & 2 & 10 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
2 & 2 & 2 & 10 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & 10 & 2 & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & 2 & \bar{2} & 2 & 2 & 10 & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 10 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
\bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 2 & 10 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 10 & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 10 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 \\
\bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & 10 & 2 & 2 & \bar{2} & 2 & \bar{2} \\
\bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & 10 & \bar{2} & 2 & \bar{2} & 2 \\
2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 10 & 2 & 2 & 2 \\
\bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & 10 & 2 & 2 \\
\bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 10 & 2 \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 2 & 10
\end{bmatrix}$$

The corresponding angle between lines is $arccos(\frac{1}{5})$.

$$H_0^t H_0 + H_1^t H_1 = \begin{bmatrix} 10 & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 \\ 2 & 10 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} \\ 2 & 2 & 10 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\ 2 & 2 & 2 & 10 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\ 2 & \bar{2} & 2 & \bar{2} & 10 & 2 & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} \\ \bar{2} & 2 & \bar{2} & 2 & 2 & 10 & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\ 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 10 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\ \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 2 & 10 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 \\ 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 10 & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} \\ 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 10 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 \\ \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & 10 & 2 & 2 & \bar{2} & 2 & \bar{2} \\ \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & 10 & \bar{2} & 2 & \bar{2} & 2 \\ 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 10 & 2 & 2 & 2 \\ \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & 10 & 2 & 2 \\ \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 10 & 2 \\ 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 2 & 10 \end{bmatrix}$$

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Sixteen Equiangular Lines in $\mathbb{R}^{10}$.

Sixteen Equiangular Lines in $\mathbb{R}^{10}$.

$$\begin{bmatrix} H_0 \\ \hline H_2 \end{bmatrix} = \left[\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
\hline
1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 \\
1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - \\
1 & - & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - \\
1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & -
\end{array}\right]$$

$$H_0^t H_0 + H_2^t H_2 = \begin{bmatrix}
10 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
2 & 10 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & 2 & 10 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 \\
2 & 2 & 2 & 10 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & 2 & \bar{2} & 2 & 10 & 2 & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & 2 & 10 & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 \\
\bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 10 & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 10 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 10 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 \\
\bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 10 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 10 & 2 & \bar{2} & 2 & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & 2 & 10 & 2 & \bar{2} & 2 & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 10 & 2 & 2 & 2 \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 10 & 2 & 2 \\
2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 10 & 2 \\
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2 & 10 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
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2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 10 & 2 & \bar{2} & 2 & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & 2 & 10 & 2 & \bar{2} & 2 & \bar{2} \\
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\end{bmatrix}
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Definition

A Hadamard matrix H is *balancedly splittable* if by suitably permuting its rows it can be transformed to

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and in this case $(\ell, a) = (\frac{n \pm \sqrt{n}}{2}, \frac{\sqrt{n}}{2})$, and the notation (n, ℓ, a) is used for $\ell = \frac{n-\sqrt{n}}{2}$ throughout.

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Delsarte, Goethals and Seidel (DGS)(1975):

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Let $X \subset \mathbb{R}^m$ be a set of unit vectors such that $|\langle v, w \rangle| = \alpha$ for all $v, w \in X, v \neq w$. If $m < \frac{1}{\alpha^2}$, then

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Nonexistence

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$$\begin{aligned}
\text{the first column} &= (\underbrace{+\cdots+}_{x \text{ rows}} \quad \underbrace{+\cdots+}_{y \text{ rows}} \quad \underbrace{+\cdots+}_{z \text{ rows}} \quad \underbrace{+\cdots+}_{w \text{ rows}})^\top, \\
\text{the } i\text{-th column} &= (+\cdots+ \quad +\cdots+ \quad -\cdots- \quad -\cdots-)^\top, \\
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\end{aligned}
$$

Then it follows that

$$
\begin{cases}
x + y + z + w &= \ell,\\
x + y - z - w &= a,\\
x - y + z - w &= a,\\
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\end{cases}
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Solving these equations yields $(x, y, z, w) = (\frac{\ell+a}{4}, \frac{\ell+a}{4}, \frac{\ell+a}{4}, \frac{\ell-3a}{4})$.

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Therefore, $\ell + a \equiv 0 \pmod 4$.
No Hadamard matrix of order $4n^2$, n odd, is balancedly splittable.

Existence

Theorem (K, Pender, Suda, DCC 2021)

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There is a balancedly splittable Hadamard matrix of order $64n^2$ for any $4n$ an order of a Hadamard matrix.

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There is a balancedly splittable Hadamard matrix of order $64n^2$ for any $4n$ an order of a Hadamard matrix.

The construction consists of building patiently nine submatrices

$$\begin{bmatrix} G & F & -F \\ E & A & B \\ -E & B & A \end{bmatrix}.$$

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Summary

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- There is a balancedly splittable Hadamard matrix of order $64n^2$ for any $4n$ an order of a Hadamard matrix. Case of $n = 12 = 4(3)$ leading to order $576 = 64(3)^2$.
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- There is a balancedly splittable Hadamard matrix of order $64n^2$ for any $4n$ an order of a Hadamard matrix. Case of $n = 12 = 4(3)$ leading to order $576 = 64(3)^2$.
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- The most important Hadamard matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix}$$

- Auxiliary matrices:

$$c_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad c_1 = \begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix}$$

The construction

The construction

S1: **Form the block Barker sequence**

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-
$$(c_0, c_1)$$
is a block Barker sequence with block autocorrelation 0

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S2: **Form the block Golay sequence**

The construction

S1: **Form the block Barker sequence**

- $$(c_0, c_1)$$
  is a block Barker sequence with block autocorrelation 0

S2: **Form the block Golay sequence**

- The sequences
  $$(c_0, c_1, c_1) \qquad (c_0, c_1, -c_1)$$
  form a block Golay pair with sum of autocorrelation 0.

The construction

S1: **Form the block Barker sequence**

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S3: **Form two block circulant matrices**

The construction

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S3: **Form two block circulant matrices**

- bcirc$(c_0 c_1 c_1)$ bcirc$(c_0 c_1 \bar{c}_1)$
  form a block complementary pair with block autocorrelation 0

The construction

S1: **Form the block Barker sequence**

- $$(c_0, c_1)$$

  is a block Barker sequence with block autocorrelation 0

S2: **Form the block Golay sequence**

- The sequences

  $$(c_0, c_1, c_1) \qquad (c_0, c_1, -c_1)$$

  form a block Golay pair with sum of autocorrelation 0.

S3: **Form two block circulant matrices**

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- Form the matrices

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A=bcirc($c_0 c_1 c_1$), B=bcirc($c_0 c_1 \bar{c}_1$)

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A=bcirc$(c_0 c_1 c_1)$, B=bcirc$(c_0 c_1 \bar{c}_1)$

Then the matrix

$$\Theta = \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \left(\begin{array}{cccccc|cccccc}
1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
- & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
\hline
1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)$$

And

$$\Theta\Theta^t = (2)\left(\begin{array}{cccccc|cccccc}
6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
\bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 \\
\hline
2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6
\end{array}\right)$$

$$
\left(\begin{array}{cccc|cccccc|cccccc}
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\hline
* & * & * & * & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
* & * & * & * & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
* & * & * & * & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
* & * & * & * & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
* & * & * & * & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
* & * & * & * & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
\hline
* & * & * & * & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
* & * & * & * & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
* & * & * & * & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
* & * & * & * & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
* & * & * & * & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
* & * & * & * & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)
$$

$$
\left(\begin{array}{c|cccccc|cccccc}
G & \multicolumn{6}{c|}{F} & \multicolumn{6}{c}{-F} \\ \hline
 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
E & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\ \hline
 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
-E & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)
$$

$$
\left(\begin{array}{cccc|cccccc|cccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
\hline
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
\hline
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)
$$

Balancedly multi-splittable Hadamard matrices

A balanced multi-splitted Hadamard matrix of order 16

A balanced multi-splitted Hadamard matrix of order 16

$$
\left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 \\
1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
1 & - & 1 & - & - & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 \\
1 & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - \\
1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & - \\
1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - \\
1 & - & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & - & 1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1
\end{array}\right]
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\left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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\end{array}\right]
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\end{array}\right]
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\left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
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1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 \\
1 & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - \\
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1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - \\
1 & - & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & - & 1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1
\end{array}\right]
$$

An Orthogonal Array; OA(5,4) on $\{1, 2, 3, 4\}$

An Orthogonal Array; OA(5,4) on $\{1, 2, 3, 4\}$ is a $4^2 \times 5$ matrix on $\{1, 2, 3, 4\}$ alphabets.

An Orthogonal Array; OA(5,4) on $\{1,2,3,4\}$ is a $4^2 \times 5$ matrix on $\{1,2,3,4\}$ alphabets.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 4 & 4 & 4 & 4 \\ 2 & 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 & 3 \\ 2 & 3 & 4 & 1 & 2 \\ 2 & 4 & 3 & 2 & 1 \\ 3 & 1 & 3 & 4 & 2 \\ 3 & 3 & 1 & 2 & 4 \\ 3 & 4 & 2 & 1 & 3 \\ 3 & 2 & 4 & 3 & 1 \\ 4 & 1 & 4 & 2 & 3 \\ 4 & 4 & 1 & 3 & 2 \\ 4 & 2 & 3 & 1 & 4 \\ 4 & 3 & 2 & 4 & 1 \end{bmatrix}$$

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A normalized Hadamard matrix H_4 :

$$\left[\begin{array}{c|ccc} \mathbf{1} & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{array}\right]$$

$$
\left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - \\
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1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
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Hadamard matrices related to projective planes

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