Additive Sums of Shifted Ternary Divisor Function

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Talk Overview

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Introduction

- The Riemann zeta function $\zeta(s)$, defined by

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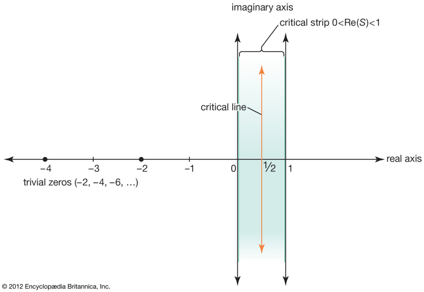
- **Riemann Hypothesis**:
  All the nontrivial zeros of the Riemann zeta function are complex numbers with real part $1/2$.



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- More precisely, one can prove that

$$\zeta(1/2+it)=\mathcal{O}(t^{\epsilon}) \iff I_k(T)=\mathcal{O}\left(T^{1+\epsilon}\right), \quad \forall k\in\mathbb{N}.$$

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- In 1998, Keating and Snaith [12] proved the asymptotic formula $I_k(T) \sim c_k\, T(\log T)^{k^2}$, where

$$c_k = \frac{g_k\, a_k}{(k^2)!},$$

with

$$g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}, \qquad a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\,\Gamma(k)}\right)^2 p^{-m}.$$

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$$D_k(X,h) = \sum_{n \le X} \tau_k(n)\tau_k(n+h), \quad h \in \mathbb{N} \neq 0,$$

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- Let us consider the fourth moment of the Riemann zeta function $I_2(T)$. In proving (1) and (2), Ingham and Heath-Brown used asymptotic formula for $D_2(X,h)$.
- Knowing a sharp error term for the correlation sum $D_3(X,h)$ also leads to an asymptotic formula for the sixth moment of the Riemann zeta function $I_3(T)$. This problem was first studied by Conrey–Gonek [2], and later Ng [16] showed that one can in fact obtain an asymptotic formula.

Introduction

- In this work, we are interested in sums of the form

$$D_{\mathcal{I},\mathcal{J}}(X,h)=\sum_{n\le X}\tau_{\mathcal{I}}(n)\tau_{\mathcal{J}}(n+h),$$

  where $\mathcal{I}=\{a_1,\dots,a_k\}$ and $\mathcal{J}=\{b_1,\dots,b_\ell\}$ are multi-sets of complex numbers which we refer to as shifts, and $\tau_{\mathcal{I}}$ is the generalized shifted divisor function given by

$$\tau_{\mathcal{I}}(n)=\sum_{n=n_1n_2\cdots n_k} n_1^{-a_1}n_2^{-a_2}\cdots n_k^{-a_k}.$$

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Survey of Unconditional Results

Recall the expression for $D_2(X,h)$ given by

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- In 1999, Meurman [14] improved the error term to

$$O\Big((x(x+h))^{1/3+\epsilon} + (x(x+h))^{1/4}x^{\epsilon}\min\Big(x^{1/4},\, h^{1/8+\alpha/2}\Big)\Big),$$

where $\alpha > 0$ is a constant satisfying $|\rho_j(n)| \le n^{\alpha}|\rho_j(1)|$.

Survey of Unconditional Results

Moving to the general case, for $k, l \in \mathbb{N}$, we set

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- In 2018, Topacogullari [19] proved $D_{k,2}(X,h) = XP_{k,h}(\log X) + \mathcal{O}\big(X^{1-\frac{4}{15k-9}+\epsilon} + X^{56/57+\epsilon}\big)$, for $k \ge 4$ and positive integers h satisfying $h \ll X^{15/19}$.

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However, no unconditional results are known beyond that. For example, we do not have an unconditional asymptotic formula for $D_{3,3}(X,h)$.

There are unconditional upper and lower bounds for $D_{k,l}(x,h)$.

- Daniel [3]: $D_{k,k}(x,h) \ll_k \prod_{p|h}\left(1 - \frac{(k-1)^2}{p}\right) x(\log x)^{2k-2}$.
- Ng and Thom [17]:

$$\frac{1}{2^{2k-2}}\frac{c_k}{((k-1)!)^2} x(\log x)^{2k-2}\left(1 + O_k\Big(\frac{\log\log h}{\log x}\Big)\right) \le D_{k,k}(x,h).$$

Conditional Results for $D_3(X, h)$

In 2023, Nguyen [18] studied

$$D_3(X, h) = \sum_{n \leq X} \tau_3(n)\tau_3(n+h),$$

and established conditional results upon the following conjecture regarding the averaged level of distribution for $\tau_k(n)$ in arithmetic progressions.

Conjecture 1 (Nguyen, 2023)

Let $\epsilon > 0$. Then, for any $k \geq 1$, we have, uniformly in $1 \leq h \leq X^{\frac{k-1}{k}}$, the upper bound

$$\sum_{q \leq X^{\frac{k-1}{k}}} \left| \sum_{\substack{n \leq X \\ n \equiv h (mod\ q)}} \tau_k(n) - \frac{1}{\varphi\left(\frac{q}{(h,q)}\right)} \sum_{\substack{n \leq X \\ \left(n, \frac{q}{(h,q)}\right) = 1}} \tau_k(n) \right| \ll_\epsilon X^{\frac{1}{2}+\epsilon},$$

as $X \to \infty$, where the implied constant is independent of h and only depends on ϵ.

Conditional Results for $D_3(X, h)$

- Assuming the Conjecture 1 for $k = 3$, one of the results established in [18] gives an asymptotic formula for $D_3(X, h)$ with lower order main terms and power savings in the error term.

Conditional Results for $D_3(X, h)$

- Assuming the Conjecture 1 for $k = 3$, one of the results established in [18] gives an asymptotic formula for $D_3(X, h)$ with lower order main terms and power savings in the error term.
- **Our goal**: extend the result as mentioned earlier of Nguyen by establishing an asymptotic formula for the shifted ternary additive divisor sum $D_{\mathcal{I},\mathcal{J}}(X, 1)$ where $\mathcal{I} = \{0, 0, 0\}$ and $\mathcal{J} = \{\beta_1, \beta_2, \beta_3\}$ with $|\beta_i| \leq \delta$, for $1 \leq i \leq 3$, for some positive constant δ.

Talk Overview

Statements of Results

Our main result assumes an upper bound for the averaged level of distribution of $\tau_{\mathcal{J}}(n)$ in arithmetic progressions.

Conjecture 2 (Vo, 2024)

Let $\epsilon > 0$, and $\mathcal{J} = \{\beta_1, \beta_2, \beta_3\}$ be a multi-set of distinct complex numbers. Suppose that $|\beta_i| \leq \delta$, for all $i \in \{1, 2, 3\}$, for some positive constant δ. Then

$$\sum_{q \leq X^{\frac{2}{3}}} \left| \sum_{\substack{n \leq X \\ n \equiv 1 (mod\ q)}} \tau_{\mathcal{J}}(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \leq X \\ (n,q)=1}} \tau_{\mathcal{J}}(n) \right| \ll_{\epsilon,\delta} X^{\frac{1}{2}+\epsilon},$$

as $X \to \infty$, where the implied constant depends only on ϵ and δ.

Statements of Results

We now state the main result.

Theorem 1 (Vo, 2024)

Assume Conjecture 2. Let $\epsilon > 0$, $\mathcal{I} = \{0,0,0\}$, *and* $\mathcal{J} = \{\beta_1, \beta_2, \beta_3\}$ *with* $\beta_i \neq \beta_j$ *for all* $i \neq j$ *and* $0 \neq |\beta_i| \leq \delta$ *for all* $i, j \in \{1,2,3\}$ *for some* $0 < \delta < \frac{5}{66}$. *As* $X \to \infty$, *we have*

$$D_{\mathcal{I},\mathcal{J}}(X,1) = Main_{\mathcal{I},\mathcal{J}}(X) + \mathcal{O}\left(X^{0.924+\delta+\epsilon}\right).$$

Statements of Results

Upon computing the residues by Maple in the above theorem, we obtain the following corollary.

Corollary 1 (Vo, 2024)

$$
\begin{aligned}
Main_{\mathcal{I},\mathcal{J}}(X) = 3\sum_{j=1}^{3}\Bigg(&\frac{(X+1)^{1-\beta_j}}{1-\beta_j}\prod_{i=1;\ i\neq j}^{3}\zeta(1-\beta_j+\beta_i)\Bigg(\gamma c_1+\left(\gamma^2+\frac{\log(X)\gamma}{3}\right)c_0\\
&+\left(-c_2-\frac{\log(X)}{3}c_1+\left(-\gamma(1)-\frac{\log^2(X)}{18}\right)c_0\right)\\
&+\left(\frac{2\log(X)}{3}c_1+\frac{2\left(\log(X)\gamma+\frac{\log^2(X)}{3}\right)}{3}c_0\right)+\left(d_4+\left(\gamma+\frac{\log(X)}{3}\right)d_2\right)\Bigg)\\
-3\sum_{j=1}^{3}\Bigg(&\frac{X^{1-\beta_j}}{(1-\beta_j)^2}\prod_{i=1;\ i\neq j}^{3}\zeta(1-\beta_j+\beta_i)\\
&\times\left(\gamma A_j^{(2)}(0,1-\beta_j)+\frac{1}{3}\log X A_j^{(2)}(0,1-\beta_j))+\frac{d}{dw_2}\left(A_j^{(2)}(w_1,1-\beta_j)\right)\Big|_{w_1=0}\right)\Bigg)\\
+\sum_{j=1}^{3}\Bigg(&\frac{X^{1-\beta_j}}{(1-\beta_j)^3}\prod_{i=1;\ i\neq j}^{3}\zeta(1-\beta_j+\beta_i)A_j^{(3)}(1-\beta_j,1-\beta_j)\Bigg).
\end{aligned}
$$

Talk Overview

Lemmas

We start with the following combinatorial lemma, which is a lemma of Hooley [9] for $\tau_3(n)$.

Lemma 1

If $n \le X$, then

$$\tau_3(n) = \sum_{l_1 l_2 l_3 = n} 1 = 3\Sigma_1(n) - 3\Sigma_2(n) + \Sigma_3(n),$$

where

$$\Sigma_1(n) = \sum_{\substack{l_1 l_2 l_3 = n \\ l_1 l_2 \le X^{2/3};\ l_1 \le X^{1/3}}} 1,$$

$$\Sigma_2(n) = \sum_{\substack{l_1 l_2 l_3 = n \\ l_1 l_2 \le X^{2/3};\ l_1, l_3 \le X^{1/3}}} 1,$$

$$\Sigma_3(n) = \sum_{\substack{l_1 l_2 l_3 = n \\ l_1, l_2, l_3 \le X^{1/3}}} 1.$$

Lemmas

Next, we consider a Dirichlet series generated by $\tau_{\mathcal{J}}(n)$.

Lemma 2

Consider $\mathcal{J} = \{\beta_1, \beta_2, \beta_3\}$*. For any* $h \geq 1$ *and* $\Re(s) > 1$*, we have*

$$\sum_{(n,h)=1} \frac{\tau_{\mathcal{J}}(n)}{n^s} = \zeta(s+\beta_1)\zeta(s+\beta_2)\zeta(s+\beta_3) \times \prod_{p|h} \left(1 - \frac{1}{p^{s+\beta_1}}\right)\left(1 - \frac{1}{p^{s+\beta_2}}\right)\left(1 - \frac{1}{p^{s+\beta_3}}\right).$$

Lemmas

Next, we consider the multidimensional Perron's formula [1].

Lemma 3

Let $f : \mathbb{N}^k \to \mathbb{C}$ be an arithmetic function in k-variables, and let $(\sigma_{a_1}, \ldots, \sigma_{a_k})$ be the k-tuple of the abscissas of absolute convergence of the associated Dirichlet series

$$F(w_1, \ldots, w_k) = \sum_{l_1, \ldots, l_k = 1}^{\infty} \frac{f(l_1, \ldots, l_k)}{l_1^{w_1} \cdots l_k^{w_k}}.$$

We have for non-integral values $x_1, \ldots, x_k \geq 1$, $c_1 > \max(0, \sigma_{a_1}), \ldots, c_k > \max(0, \sigma_{a_k})$, and $T_1 \geq 1, \cdots, T_k \geq 1$,

$$\left| \sum_{l_1 \leq x_1, \ldots, l_k \leq x_k} f(l_1, \ldots, l_k) - \frac{1}{(2\pi i)^k} \int_{c_1 - iT_1}^{c_1 + iT_1} \cdots \int_{c_k - iT_k}^{c_k + iT_k} F(w_1, \ldots, w_k) \frac{x_1^{w_1} \cdots x_k^{w_k}}{w_1 \cdots w_k} dw_k \cdots dw_1 \right|$$

$$\leq x_1^{c_1} \cdots x_k^{c_k} \sum_{l_1, \ldots, l_k = 1}^{\infty} \frac{|f(l_1, \ldots, l_k)|}{l_1^{c_1} \cdots l_k^{c_k}} Q\left(\frac{1}{\max(\pi T_1 |\log(x_1/l_1)|, 1)}, \ldots, \frac{1}{\max(\pi T_k |\log(x_k/l_k)|, 1)} \right),$$

where

$$Q(X_1, \ldots, X_k) = (X_1 + 1) \cdots (X_k + 1) - 1.$$

Talk Overview

Sketch of Proof

- **Step 1:** Apply Lemma 1 for $\tau_{\mathcal{I}}(n)$ in $D_{\mathcal{I},\mathcal{J}}(X,1)$, we have

$$D_{\mathcal{I},\mathcal{J}}(X,1) = 3\Sigma_{11}(X) - 3\Sigma_{21}(X) + \Sigma_{31}(X).$$

Note: The procedure is similar for the three terms, so we only need to consider the first term $\Sigma_{11}(X)$.
Now, making a change of variable in the l_3 sum, we have

$$\begin{aligned}\Sigma_{11}(X) &= \sum_{l_1 \leq X^{1/3}} \sum_{l_2 \leq \frac{X^{2/3}}{l_1}} \sum_{l_3 \leq \frac{X}{l_1 l_2}} \tau_{\mathcal{J}}(l_1 l_2 l_3 + 1) \\ &= \sum_{l_1 \leq X^{1/3}} \sum_{l_2 \leq \frac{X^{2/3}}{l_1}} \sum_{\substack{n \leq X+1 \\ n \equiv 1 (l_1 l_2)}} \tau_{\mathcal{J}}(n).\end{aligned}$$

Sketch of Proof

- **Step 2:** Implement the Conjecture 2 as previously outlined.

$$\sum_{\substack{n \le Y \\ n \equiv 1 (mod\ q)}} \tau_{\mathcal{J}}(n) = \frac{1}{\varphi(q)} \sum_{\substack{n \le Y \\ (n,q)=1}} \tau_{\mathcal{J}}(n) + E(Y;1,q),$$

where

$$\sum_{q \le Y^{2/3}} E(Y;1,q) \ll Y^{1/2+\epsilon}.$$

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$$\sum_{q\leq Y^{2/3}} E(Y;1,q) \ll Y^{1/2+\epsilon}.$$

- **Step 3:** Employ the Dirichlet series by utilizing the $\tau_{\mathcal{J}}(n)$ in Lemma 2 alongside truncated Perron's formula.

$$\sum_{\substack{n\leq Y\\ n\equiv 1 (mod\ q)}} \tau_{\mathcal{J}}(n) = \frac{1}{\varphi(q)} Res_{\substack{s=1-\beta_1\\ s=1-\beta_2\\ s=1-\beta_3}} \frac{Y^s}{s}\zeta(s+\beta_1)\zeta(s+\beta_2)\zeta(s+\beta_3)f_q(s)$$

$$+\mathcal{O}\left(\frac{1}{\varphi(q)}Y^{3/4+\delta+\delta^2+\epsilon}\right) + E(Y;1,q),$$

where

$$f_q(s) = \prod_{p|q}\left(1-\frac{1}{p^{s+\beta_1}}\right)\left(1-\frac{1}{p^{s+\beta_2}}\right)\left(1-\frac{1}{p^{s+\beta_3}}\right).$$

Sketch of Proof

- **Step 4:** Apply the previous step to the last sum of $\Sigma_{11}(X)$.

$$\begin{aligned}
&\Sigma_{11}(X)\\
&\quad= \sum_{l_1 \le X^{1/3}} \sum_{l_2 \le \frac{X^{2/3}}{l_1}} \sum_{\substack{n \le X+1 \\ n \equiv 1 (l_1 l_2)}} \tau_{\mathcal{J}}(n)\\
&\quad= \sum_{l_1 \le X^{1/3}} \sum_{l_2 \le \frac{X^{2/3}}{l_1}} \left(\frac{1}{\varphi(l_1 l_2)} Res_{\substack{s=1-\beta_1 \\ s=1-\beta_2 \\ s=1-\beta_3}} \frac{(X+1)^s}{s} F(s) \right.\\
&\qquad\qquad \left. + \mathcal{O}\left(\frac{1}{\varphi(l_1 l_2)} (X+1)^{3/4+\delta+\delta^2+\epsilon} \right) + E\left(X+1; 1, l_1 l_2\right) \right)\\
&\quad= \sum_{l_1 \le X^{1/3}} \sum_{l_2 \le \frac{X^{2/3}}{l_1}} \frac{1}{\varphi(l_1 l_2)} Res_{\substack{s=1-\beta_1 \\ s=1-\beta_2 \\ s=1-\beta_3}} \frac{(X+1)^s}{s} \zeta(s+\beta_1)\zeta(s+\beta_2)\zeta(s+\beta_3) f_{l_1 l_2}(s)\\
&\qquad + \mathcal{O}\left(X^{3/4+\delta+\delta^2+\epsilon} \right),
\end{aligned}$$

Sketch of Proof

- **Step 5:** Analyze and compute the main term of $\Sigma_{11}(X)$.

$$
\begin{aligned}
&M^{(1)}_{\mathcal{I},\mathcal{J}}(X) \\
&= \sum_{l_1 \le X^{1/3}} \sum_{l_2 \le \frac{X^{2/3}}{l_1}} \frac{1}{\varphi(l_1 l_2)} Res_{\substack{s=1-\beta_1\\ s=1-\beta_2\\ s=1-\beta_3}} \frac{(X+1)^s}{s} \zeta(s+\beta_1)\zeta(s+\beta_2)\zeta(s+\beta_3) f_{l_1 l_2}(s) \\
&= 3 \sum_{j=1}^{3} \left(\frac{(X+1)^{1-\beta_j}}{1-\beta_j} \prod_{\substack{i=1\\ i\neq j}}^{3} \zeta(1-\beta_j+\beta_i) \sum_{l_1 \le X^{1/3}} \sum_{l_2 \le \frac{X^{2/3}}{l_1}} \frac{f_{l_1 l_2}(1-\beta_j)}{\varphi(l_1 l_2)} \right),
\end{aligned}
$$

with the indispensable assistance of multidimensional Perron's formula in Lemma 3, and the subconvexity bound for $\zeta(s)$ due to Weyl [20] and Hardy-Littlewood [13]: for $\lambda \le \frac{1}{6}$, $1/2 \le \sigma \le 1$ and any $\epsilon > 0$, we have

$$|\zeta(\sigma+it)| \ll_\epsilon (1+|t|)^{2\lambda(1-\sigma)+\epsilon}.$$

Talk Overview

Conclusion

- We investigated into the shifted convolution sum

$$D_{\mathcal{I},\mathcal{J}}(X,1) = \sum_{n \le X} \tau_{\mathcal{I}}(n)\tau_{\mathcal{J}}(n+1),$$

where $\mathcal{I} = \{0,0,0\}$ and $\mathcal{J} = \{\beta_1,\beta_2,\beta_3\}$ with $|\beta_i|$ being sufficiently small for $i \in \{1,2,3\}$.

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- In a joint work in progress with Dr. Alia Hamieh, we aim at extending our results to include the case where $\mathcal{I} = \{\alpha_1, \alpha_2, \alpha_3\}$ with non-zero complex shifts α_i that are sufficiently small in magnitude for $i \in \{1,2,3\}$.

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- **Future work**: extend the methods of Nguyen [18] to obtain an asymptotic formula for the more general shifted convolution sum

$$D_{\mathcal{I},\mathcal{J}}(X,h) = \sum_{n \le X} \tau_{\mathcal{I}}(n)\tau_{\mathcal{J}}(n+h),$$

  where $1 \le h \le X^{\frac{k-1}{k}}$, $\mathcal{I} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and $\mathcal{J} = \{\beta_1, \beta_2, \ldots, \beta_k\}$.

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Thanks for your attention!

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