Bounds on the Number of Solutions to Thue Equations

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Land Acknowledgment

The University of Calgary, located in the heart of Southern Alberta, both acknowledges and pays tribute to the traditional territories of the peoples of Treaty 7, which include the Blackfoot Confederacy (comprised of the Siksika, the Piikani, and the Kainai First Nations), the Tsuut'ina First Nation, and the Stoney Nakoda (including Chiniki, Bearspaw, and Goodstoney First Nations). The City of Calgary is also home to the Métis Nation of Alberta, Districts 5 and 6.

Intro to Thue Equations

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- The homogeneity condition is also necessary: if $F(x,y) = x^6 + y^3$, then any integer pair of the form $(n, -n^2)$ will be a solution to $|F(x,y)| \leqslant h$.

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- What are the (integer) solutions to $|F(x, y)| \leqslant h$?
- How many solutions are there to $|F(x, y)| \leqslant h$?
- On which features of $F(x, y)$ and h do the number of solutions depend?

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- E.g. if $f(X) = X^6 - 3X^4 + 1$, then

$$y^6 f\left(\frac{x}{y}\right) = y^6\left(\left(\frac{x}{y}\right)^6 - 3\left(\frac{x}{y}\right)^4 + 1\right) = F(x, y)$$

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- Hence, the condition that $|F(x, y)| = 1$ is equivalent to the condition that $x - \alpha_i y$ is a unit for each i.

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- Baker's method gives lower bounds on the left-hand side, and hence, an upper bound on u_2.
- Those bounds can be traced back to bounds on x and y.

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Theorem (Baker, 1968)

Suppose that $F(x, y)$ has degree n and $\kappa > n$. Then any $x, y \in \mathbb{Z}$ with $|F(x, y)| \leqslant h$ has

$$\max(|x|, |y|) \leqslant C_{F,\kappa} h^{(\log h)^{\kappa - 1}}$$

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- Compute $C_{F,\kappa}$.
- Test all pairs $(x,y) \in \mathbb{Z}^2$ satisfying $\max(|x|,|y|) \leqslant C_{F,\kappa} e^{(\log h)^{\kappa}}$ to see if $|F(x,y)| \leqslant h$.

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Question

Is this what the growth rate of $N(F,h)$ actually looks like?

Aside

"Eliminating" h

Observe that $|x^5 + 3x^4y - y^5| \leqslant h$ if and only if

$$\left|\left(\frac{x}{h^{1/5}}\right)^5 + 3\left(\frac{x}{h^{1/5}}\right)^4\left(\frac{y}{h^{1/5}}\right) - \left(\frac{y}{h^{1/5}}\right)^5\right| \leqslant 1.$$

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Fact

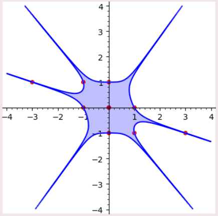
$|F(x, y)| \leqslant h$ *if and only if*

$$\left| F\left(\frac{x}{h^{1/n}}, \frac{y}{h^{1/n}}\right) \right| \leqslant 1.$$

Geometric View of $|F(x, y)| \leqslant h$

A Picture

$|F(x, y)| \leqslant h$ corresponds to a region of the xy-plane:



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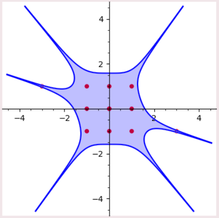
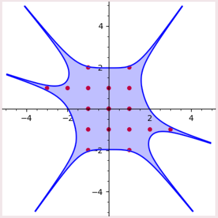


$|x^5 + 3x^4y - y^5| \leqslant 10$



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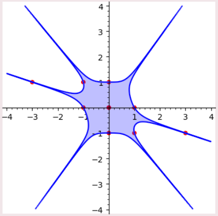

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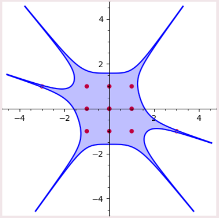

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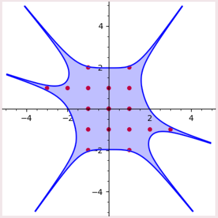

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Computing $N(F, h)$

Some values of $N(F, h)$ for $F(x, y) = x^5 + 3x^4y - y^5$:

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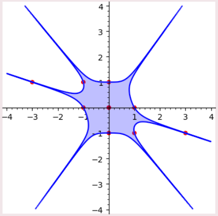

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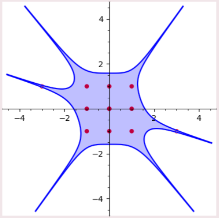

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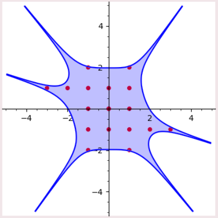

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h	1	10	30
$N(F, h)$	9	11	17

Exploring Dependence on h

$N(F, h)$ and volume

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Exploring dependence on h

Theorem (Mahler, 1934)

Let

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Next Steps

Now we aim to estimate $N(F, 1) = \#\{(x, y) \in \mathbb{Z}^2 : |F(x, y)| = 1\}$.

Important Features of $F(x, y)$

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Question

How does $N(F, 1)$ depend on n, s, and H?

The Big Idea: Rational Approximation

Start with a solution, $(p, q) \in \mathbb{Z}^2$ with $q \neq 0$, so that

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Every solution (p, q) to the Thue equation $|F(x, y)| = 1$ with $q \neq 0$ yields a good rational approximation $\frac{p}{q}$ to a root α of $f(X) = F(X, 1)$.

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Why do we care about rational approximations of algebraic numbers?

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Note

Pairs of integers (p, q) are not in bijection with rational numbers $\frac{p}{q}$. Sometimes, we will count primitive solutions, i.e. those with $\gcd(p, q) = 1$.

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- If $|F(p, q)| = 1$, then $\frac{p}{q}$ is close to a root of $f(X) := F(X, 1)$.
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Lemma (Descartes, 1637)

If $g(x) \in \mathbb{R}[x]$ has $s + 1$ nonzero summands, then $g(x)$ has no more than $2s + 1$ real roots.

Exploring $N(F, 1)$

Previous Facts

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We expect the number of rational approximations per root to be absolutely bounded.

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Conclusion

We expect there to be no more than a constant times s solutions to $|F(x, y)| = 1$.

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The symbol $\ll$ means "(is) no more than a constant times."

Conclusion (rephrased)

We expect there to be $\ll s$ solutions to $|F(x,y)| = 1$.

A Conjecture and Theorem of Mueller and Schmidt

The Pieces

Recall that we expect:

$$N(F, h) \approx h^{2/n} \cdot N(F, 1).$$
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Theorem (Saradha and Sharma, 2017)

$$N(F,h) \ll s e^{\Phi} h^{2/n}(1 + \log h^{1/n})$$

where Φ *measures the "sparsity" of* $F(x,y)$ *and satisfies* $(\log s)^3 \leqslant e^{\Phi} \ll s$.

Weak Assumptions on s and h

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If $n \geqslant s(\log s)^3$, then

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Theorem (Akhtari and Bengoechea, 2020)

If h is small relative to the discriminant of $F(x, y)$, then

$$N(F, h) \ll s(\log s)\min\left(1, \frac{1}{\log n - \log s}\right).$$

Picking Values for s and h

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Theorem (Thomas, 2000)

If $F(x,y) = ax^n + bx^k y^{n-k} + cy^n$, *there are no more than* $C_1(n)$ *solutions to* $|F(x,y)| = 1$ *where* $C_1(n)$ *is defined by*

n	6	7	8	9	10-11	12-16	17-37	$\geqslant 38$
$C_1(n)$	136	86	96	62	72	60	56	48

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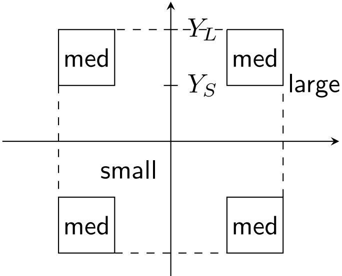

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- Technique: archimedean Newton polygons

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Moral

There's a set of $\ll s$ algebraic numbers so that any solution to $|F(x, y)| \leqslant h$ with $x, y > Y_S$ gives a rational number $\frac{x}{y}$ or $\frac{y}{x}$ which is close to one of those algebraic numbers.

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- Enumerate the medium solutions which satisfy the above inequality, and order them so that

$$Y_S < y_0 \leqslant y_1 \leqslant \cdots \leqslant y_t < Y_L.$$

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- Use the fact that if $\frac{x_i}{y_i}$ and $\frac{x_{i+1}}{y_{i+1}}$ are close to α, they are close to each other:

$$\begin{aligned}\frac{K}{y_i^{n/s}} &> \left|\frac{x_i}{y_i} - \frac{x_{i+1}}{y_{i+1}}\right| \\ &= \left|\frac{x_i y_{i+1} - x_{i+1} y_i}{y_i y_{i+1}}\right| \\ &\geqslant \frac{1}{y_i y_{i+1}}\end{aligned}$$

implying that $y_{i+1} > \frac{y_i^{\frac{n}{s}-1}}{K}$.

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  implying that $y_{i+1} > \frac{y_i^{\frac{n}{s}-1}}{K}$.
- This is known as The Gap Principle.

Counting

Counting with Gaps

Using $y_{i+1} > \frac{y_i^{\frac{n}{s}-1}}{K}$ together with $Y_S < y_0 \leqslant y_1 \leqslant \cdots \leqslant y_t < Y_L$, we can find bounds on t.

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Lemma (K., 2023)

If $n \geqslant 3s$ and there are $t+1$ medium solutions associated to α, then

$$t \leqslant \frac{\log\left[\frac{\log Y_L K^{-1/(\frac{n}{s}-2)}}{\log Y_S K^{-1/(\frac{n}{s}-2)}}\right]}{\log\left(\frac{n}{s}-1\right)}$$

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Something more useful

Reducing the above constants into terms of n, s, h, H, using $n \geqslant 3s$, and applying the fact that there are $\ll s$ roots α that we need to care about, we find...

Counting Medium Solutions

Theorem (K., 2023)

The number of primitive medium solutions to $|F(x,y)| \leqslant h$ when $n \geqslant 3s$ is

$$\ll s\left(1+\log\left(s+\frac{\log h}{\max(1,\log H)}\right)\right).$$

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Recall:

Conjecture

If $h \leqslant H^{1-\frac{s}{n}-\rho}$, then the number of primitive solutions to $|F(x,y)| \leqslant h$ is bounded by a function only of s and ρ.

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Challenges

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Theorem (Saradha-Sharma, 2017)

When $n > 4se^{2\Phi}$, the number of primitive small solutions to $|F(x,y)| \leqslant h$ is

$$\ll se^{\Phi}h^{2/n},$$

where Φ measures the "sparsity" of F and satisfies $\log^3 s \leqslant e^{\Phi} \ll s$.

The Big Picture

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Specific Cases

When we have explicit bounds on $N(F, h)$, my improvements do yield tighter bounds.

Explicit Bounds for Trinomials

Theorem (Thomas, 2000)

If $F(x,y) = ax^n + bx^ky^{n-k} + cy^n$, there are no more than $C_1(n)$ solutions to $|F(x,y)| = 1$ where $C_1(n)$ is defined by

n	6	7	8	9	10-11	12-16	17-37	$\geqslant 38$
$C_1(n)$	136	86	96	62	72	60	56	48

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Question

Is this a good bound?

Trinomial Computations

H	1	2	3	4	5	6	7	8	9	10	$\cdots$	16
$n = 6$	8	6	8	8	6	6	6	6	8	6	$\cdots$	12
$n = 7$	8	6	8	8	6	6	6	6	8	6	$\cdots$	8
$n = 8$	8	6	8	8	6	6	6	6	8	6	$\cdots$	12
$n = 9$	8	6	8	8	6	6	6	6	8	6	$\cdots$	8
$n = 10$	8	6	8	8	6	6	6	6	8	-	$\cdots$	8
$n = 11$	8	6	8	8	6	6	6	6	8	-	$\cdots$	-
$n = 12$	8	6	8	8	6	6	6	-	-	-	$\cdots$	-
$n = 13$	8	6	8	8	6	6	-	-	-	-	$\cdots$	-
$n = 14$	8	6	8	8	6	6	-	-	-	-	$\cdots$	-
$n = 15$	8	6	8	8	6	-	-	-	-	-	$\cdots$	-
$n = 16$	8	6	8	8	6	-	-	-	-	-	$\cdots$	-
$n = 17$	8	6	8	8	-	-	-	-	-	-	$\cdots$	-

Maximum number of solutions to $|F(x, y)| = 1$ for any trinomial of height H and degree n

Thank you!

Questions?