Joint distribution of central values and orders of Sha groups of quadratic twists of an elliptic curve

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L-functions of quadratic twists of elliptic curves

For an elliptic curve $E/\mathbb{Q}$ of conductor $N = N_E$, the associated Hasse-Weil L-function is

$$L(s, E) = \sum_{n=1}^{\infty} \frac{\lambda_E(n)}{n^s} \quad \text{for} \quad \mathfrak{Re}(s) > 1,$$

where $|\lambda_E(n)| \leq d_2(n)$. This L-function extends to an entire function and satisfies

$$\Lambda(s, E) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s + \tfrac{1}{2}) L(s, E) = \epsilon_E \Lambda(1 - s, E) \quad \text{with} \quad \epsilon_E = \pm 1.$$

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For any fundamental discriminant d coprime to $2N$, E_d will stand for the quadratic twist of E by $\chi_d = (\frac{d}{\cdot})$. The associated twisted L-function is

$$L(s, E_d) = \sum_{n=1}^{\infty} \frac{\lambda_E(n)\chi_d(n)}{n^s}.$$

As $(d, N) = 1$, the conductor of E_d is Nd^2, and one knows

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By Waldspurger's theorem, $L(\frac{1}{2}, E_d) \ge 0$. As $L(\frac{1}{2}, E_d) = 0$ if $\epsilon_E(d) = -1$, we consider

$$\mathcal{E} = \{d : d \text{ is a fundamental discriminant with } (d, 2N) = 1 \text{ and } \epsilon_E(d) = 1\}.$$

About central L-values

- Values of $L(s, E_d)$ and its derivatives at $s = \frac{1}{2}$ encode deep arithmetic information (predicted by the Birch and Swinnerton-Dyer conjecture).
- (After Kolyvagin, Murty-Murty, and Bump-Friedberg-Hoffstein), there are infinitely many of these values (and their derivatives) that are non-vanishing.
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- Asymptotic for the second moment of $L(\frac{1}{2}, E_d)$ was established by Soundararajan-Young (under GRH) and Xiannan Li (unconditionally).
- Goldfeld's conjecture: for almost all $d \in \mathcal{E}$, $L(\frac{1}{2}, E_d) \neq 0$.
- Heath-Brown proved that under GRH for at least $\frac{1}{4}$ of $d \in \mathcal{E}$, $L(\frac{1}{2}, E_d) \neq 0$.
- Smith proved that under BSD (and a mild condition), Goldfeld's conjecture is true!

A conjecture of Keating-Snaith

As d ranges in $\mathcal{E}$, $\log L(\frac{1}{2}, E_d)$ shall behave like a normal random variable with mean $-\frac{1}{2}\log\log|d|$ and variance $\log\log|d|$.

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$$\#\left\{d \in \mathcal{E}, X < |d| \leq 2X : \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \in (\alpha, \beta)\right\}$$
$$= (\Psi(\alpha, \beta) + o(1))\#\{d \in \mathcal{E} : X < |d| \leq 2X\},$$

as $X \to \infty$, where

$$\Psi(\alpha, \beta) = \int_\alpha^\beta \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt,$$

and the kernel of the integral is the probability density function of a standard normal random variable (i.e., with mean 0 and variance 1).

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and the kernel of the integral is the probability density function of a standard normal random variable (i.e., with mean 0 and variance 1).
This can be seen as a generalisation of Selberg's central limit theorem asserting that

$$\frac{\log|\zeta(\frac{1}{2} + \mathrm{i}t)|}{\sqrt{\frac{1}{2}\log\log T}}$$

is approximately standard normal.

Upper and lower bounds towards the Keating-Snaith conjecture for $L(\frac{1}{2}, E_d)$

Theorem (Radziwiłł-Soundararajan, 2015 & 2023)

Unconditionally, for any fixed $V \in \mathbb{R}$, as $X \to \infty$,

$$\#\left\{d \in \mathcal{E}, 20 < |d| \le X : \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \ge V\right\}$$
$$\le (\Psi(V,\infty) + o(1))\#\{d \in \mathcal{E} : |d| \le X\}.$$

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$$\leq (\Psi(V,\infty) + o(1))\#\{d \in \mathcal{E} : |d| \leq X\}.$$

Assume GRH for all $L(s, E \otimes \chi)$ with Dirichlet characters χ. Then for any fixed (α, β),

$$\#\left\{d \in \mathcal{E}, X < |d| \leq 2X : \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \in (\alpha,\beta)\right\}$$
$$\geq \frac{1}{4}(\Psi(\alpha,\beta) + o(1))\#\{d \in \mathcal{E} : X < |d| \leq 2X\},$$

as $X \to \infty$.

- The factor $\frac{1}{4}$ coincides with the proportion of non-vanishing $L(\frac{1}{2}, E_d)$ established by Heath-Brown under GRH.
- Recently, Bui, Evans, Lester, and Pratt proved a full asymptotic for an analogue of Keating-Snaith's conjecture (with the vanishing central values assigned a weight equal to zero).

A conjecture of Radziwiłł and Soundararajan

Based on Keating-Snaith's conjecture, Radziwiłł and Soundararajan formulated the following conjecture regarding the distribution of orders of Tate-Shafarevich groups $\text{Ш}(E_d)$ of E_d.

Conjecture (Radziwiłł-Soundararajan, 2015)

Let E be given in Weierstrass form $y^2 = f(x)$ for a monic cubic integral polynomial f, and let K denote the splitting field of f over $\mathbb{Q}$. Define $c(g) \in \mathbb{N}$ so that $c(g) - 1$ is the number of fixed points of $g \in \mathrm{Gal}(K/\mathbb{Q})$, and set

$$\mu(E) = -\frac{1}{2} - \frac{1}{|G|}\sum_{g \in G} \log c(g) \quad \textit{and} \quad \sigma(E) = 1 + \frac{1}{|G|}\sum_{g\in G}(\log c(g))^2.$$

Then, as d ranges over $\mathcal{E}$, the distribution of $\log(|\text{Ш}(E_d)|/\sqrt{|d|})$ is approximately Gaussian, with mean $\mu(E)\log\log|d|$ and variance $\sigma(E)^2\log\log|d|$. Note that denoting n_K the degree of K, one has the following table of explicit values of $\mu(E)$ and $\sigma(E)^2$.

n_K	1	2	3	6
$\mu(E)$	$-\frac{1}{2} - 2\log 2$	$-\frac{1}{2} - \frac{3}{2}\log 2$	$-\frac{1}{2} - \frac{2}{3}\log 2$	$-\frac{1}{2} - \frac{5}{6}\log 2$
$\sigma(E)^2$	$1 + 4(\log 2)^2$	$1 + \frac{5}{2}(\log 2)^2$	$1 + \frac{4}{3}(\log 2)^2$	$1 + \frac{7}{6}(\log 2)^2$

What is known?

When $L(\frac{1}{2}, E_d) \neq 0$, there is an analytic correspondence of $|\text{Ш}(E_d)|$ defined by

$$S(E_d) = L(\tfrac{1}{2}, E_d)\frac{|E_d(\mathbb{Q})_{\text{tors}}|^2}{\Omega(E_d)\,\text{Tam}(E_d)},$$

where $|E_d(\mathbb{Q})_{\text{tors}}|$ denotes the order of the rational torsion group of E_d, $\Omega(E_d)$ is the real period of a minimal model for E_d, and $\text{Tam}(E_d) = \prod_p T_p(d)$ is the product of the Tamagawa numbers. (Note that under "rank zero" BSD, $|\text{Ш}(E_d)| = S(E_d)$.)

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Theorem (Radziwiłł-Soundararajan, 2015)

Unconditionally, for any fixed $V \in \mathbb{R}$, as $X \to \infty$,

$$\#\left\{d \in \mathcal{E}, 20 < |d| \le X : \frac{\log(S(E_d)/\sqrt{|d|}) - \mu(E)\log\log|d|}{\sqrt{\sigma(E)^2 \log\log|d|}} > V\right\}$$

is bounded above by $(\Psi(V,\infty) + o(1))\#\{d \in \mathcal{E} : |d| \le X\}$.
Moreover, if BSD holds for elliptic curves with analytic rank zero, then the quantity above is also an upper bound for

$$\#\left\{d \in \mathcal{E}, 20 < |d| \le X : L(\tfrac{1}{2}, E_d) \neq 0, \ \frac{\log(|\text{Ш}(E_d)|/\sqrt{|d|}) - \mu(E)\log\log|d|}{\sqrt{\sigma(E)^2 \log\log|d|}} > V\right\}.$$

A joint distribution conjecture

Conjecture (W., 2024+)

As d ranges over $\mathcal{E}$, the joint distribution of $\log L(\frac{1}{2}, E_d)$ *and* $\log(|\text{Ш}(E_d)|/\sqrt{|d|})$ *is approximately bivariate normal. More precisely,*

$$\#\Bigg\{d \in \mathcal{E}, 20 < |d| \le X : \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \in (\alpha_1, \beta_1), \frac{\log(\text{Ш}(E_d)/\sqrt{|d|}) - \mu(E)\log\log|d|}{\sqrt{\sigma(E)^2 \log\log|d|}} \in (\alpha_2, \beta_2)\Bigg\}$$

is asymptotic to $(\Xi_E(\underline{\alpha}, \underline{\beta}) + o(1))\#\{d \in \mathcal{E} : 20 < |d| \le X\}$*, as* $X \to \infty$*, where*

$$\Xi_E(\underline{\alpha}, \underline{\beta}) = \int_{(\alpha_1,\beta_1)\times(\alpha_2,\beta_2)} \frac{1}{2\pi\sqrt{\det(\mathfrak{K}_E)}} e^{-\frac{1}{2}\mathrm{v}^{\mathrm{T}}\mathfrak{K}_E^{-1}\mathrm{v}} d\mathrm{v}, \quad \mathfrak{K}_E = \begin{pmatrix} 1 & \sigma(E)^{-1} \\ \sigma(E)^{-1} & 1 \end{pmatrix}.$$

- By slightly modifying an argument of Radziwiłł-Soundararajan (2015), we showed that the conjecture is valid "from above" (i.e. $\lesssim$ holds unconditionally for sufficiently large β_1, β_2).

Towards the joint distribution

Theorem (W., 2024+)

Assume GRH for the family of twisted L-functions $L(s, E \otimes \chi)$ with all Dirichlet characters χ. For any fixed $\underline{\alpha} = (\alpha_1, \alpha_2)$ and $\underline{\beta} = (\beta_1, \beta_2)$, as $X \to \infty$,

$$\#\left\{ d \in \mathcal{E}, X < |d| \le 2X : \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \in (\alpha_1, \beta_1), \right.$$
$$\left. \frac{\log(S(E_d)/\sqrt{|d|}) - \mu(E)\log\log|d|}{\sqrt{\sigma(E)^2\log\log|d|}} \in (\alpha_2, \beta_2) \right\}$$

is greater or equal to

$$\frac{1}{4}(\Xi_E(\underline{\alpha}, \underline{\beta}) + o(1))\#\{d \in \mathcal{E} : X < |d| \le 2X\}.$$

Furthermore, suppose that BSD holds for elliptic curves with analytic rank zero. Then the above assertion is true with $S(E_d)$ being replaced by $|Ш(E_d)|$.

Consequences

- The theorem implies the early-mentioned theorem of Radziwiłł-Soundararajan.
- Also, as $X \to \infty$, we have

$$\#\left\{ d \in \mathcal{E}, X < |d| \le 2X : \frac{\log(S(E_d)/\sqrt{|d|}) - \mu(E) \log\log|d|}{\sqrt{\sigma(E)^2 \log\log|d|}} \in (\alpha_2, \beta_2) \right\}$$
$$\ge \frac{1}{4}(\Psi(\alpha_2, \beta_2) + o(1)) \#\{d \in \mathcal{E} : X < |d| \le 2X\}.$$

- Again, under BSD for elliptic curves with analytic rank zero, the above assertion is true with $S(E_d)$ being replaced by $|\text{Ш}(E_d)|$.

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Following Soundararajan (à la Selberg)

Let $x = X^{1/\log\log\log X}$, and define

$$\mathcal{P}(d;x) = \sum_{\substack{p\le x\\ p\nmid N_0}} \frac{\lambda_E(p)}{\sqrt{p}}\chi_d(p).$$

Assume GRH for $L(s,E_d)$, and let $\frac{1}{2}+i\gamma_d$ run over the non-trivial zeros of $L(s,E_d)$. If $L(\frac{1}{2},E_d)$ is non-vanishing, then for every $d\in\mathcal{E}$ with $X\le |d|\le 2X$, one has

$$\log L(\tfrac{1}{2},E_d) = \mathcal{P}(d;x) - \tfrac{1}{2}\log\log X + O\left(\log\log\log X + \sum_{\gamma_d}\log\left(1+\frac{1}{(\gamma_d\log x)^2}\right)\right),$$

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$$\log L(\tfrac{1}{2},E_d) = \mathcal{P}(d;x) - \tfrac{1}{2}\log\log X + O\Bigg(\log\log\log X + \sum_{\gamma_d}\log\left(1+\frac{1}{(\gamma_d\log x)^2}\right)\Bigg),$$

and

$$\log(S(E_d)/\sqrt{|d|}) - \mu(E)\log\log|d|$$
$$= \mathcal{P}(d;x) - \mathcal{C}(d;x) + O\Bigg((\log\log\log X)^2 + \sum_{\gamma_d}\log\left(1+\frac{1}{(\gamma_d\log x)^2}\right)\Bigg),$$

for all but at most $\ll X/\log\log\log X$ $d\in\mathcal{E}$ with $X\le|d|\le 2X$, where

$$\mathcal{C}(d;x) = \sum_{\log X\le p\le x}\left(\log T_p(d) - \frac{1}{p+1}\log c(p)\right),$$

$c(p) = 1 +$ the no. of sol. of $f(z)\equiv 0 \pmod p$, and E is given by the model $y^2 = f(z)$.

Weighted moments

Let h be a smooth even function such that $h(t) \ll (1+t^2)^{-1}$, and its Fourier transform is compactly supported in $[-1,1]$. Fix $b, c \in \mathbb{R}$. Then for any $L \geq 1$ such that $e^L \leq X^{2-10\varepsilon}$, we have

$$\sum_{d \in \mathcal{E}(\kappa,a)} (b\mathcal{P}(d;x) + c(\mathcal{P}(d;x) - \mathcal{C}(d;x)))^k \sum_{\gamma_d} h\left(\frac{\gamma_d L}{2\pi}\right) \Phi\left(\frac{\kappa d}{X}\right)$$
$$= \frac{X}{N_0} \prod_{p \nmid N_0} \left(1 - \frac{1}{p^2}\right) \widehat{\Phi}(0) \left(\frac{2\log X}{L}\widehat{h}(0) + \frac{h(0)}{2} + O(L^{-1})\right) \tag{1}$$
$$\times \left((b^2 + 2bc + c^2\sigma(E)^2)\log\log X\right)^{\frac{k}{2}} (M_k + o(1)) + O(X^{\frac{1}{2}+\varepsilon} e^{\frac{L}{4}}),$$

where the implied constants depend on b, c, and

$$M_k = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ \frac{k!}{2^{k/2}(k/2)!} & \text{if } k \text{ is even.} \end{cases}$$

is the k-th moment of the standard normal random variable.

An application of the Cramér-Wold device

Proposition

Let $\alpha_i < \beta_i$ *be real numbers, and set* $\underline{\alpha} = (\alpha_1, \alpha_2)$ *and* $\underline{\beta} = (\beta_1, \beta_2)$*. Let* $\mathcal{H}_X(\underline{\alpha}, \underline{\beta})$ *be the set of discriminants* $d \in \mathcal{E}$*, with* $X \leq |d| \leq 2X$*, such that*

$$\mathcal{Q}_1(d;X) = \frac{\mathcal{P}(d;x)}{\sqrt{\log\log X}} \in (\alpha_1, \beta_1) \quad \text{and} \quad \mathcal{Q}_2(d;X) = \frac{\mathcal{P}(d;x) - \mathcal{C}(d;x)}{\sqrt{\sigma(E)^2 \log\log X}} \in (\alpha_2, \beta_2),$$

while $L(s, E_d)$ *has no zeros* $\frac{1}{2} + \gamma_d$ *with* $|\gamma_d| \leq ((\log X)(\log\log X))^{-1}$*. Then for any* $\delta > 0$*, we have*

$$\mathcal{H}_X(\underline{\alpha}, \underline{\beta}) \geq \left(\frac{1}{4} - \delta\right)(\Xi_E(\underline{\alpha}, \underline{\beta}) + o(1))\#\{d \in \mathcal{E} : X \leq |d| \leq 2X\}.$$

- We also invoked the existence theorem of bivariate normal random variables.

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- We also invoked the existence theorem of bivariate normal random variables.
- Note that Radziwiłł and Soundararajan showed that the number of discriminants $d \in \mathcal{E}$, with $X \le |d| \le 2X$, such that

$$\sum_{|\gamma_d| \ge ((\log X)(\log\log X))^{-1}} \log\left(1 + \frac{1}{(\gamma_d \log x)^2}\right) \ge (\log\log\log X)^3$$

is $\ll X/\log\log\log X$.

Thank you:)