A Weyl-type inequality for irreducible elements in function fields, with applications

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Weyl differencing

Let us begin with the differencing process. Write $e(x) = e^{2\pi i x}$ for real x. Let $f(x) = \sum_{j=0}^{k} \alpha_j x^j \in \mathbb{R}[x]$. Weyl observed that

$$\left| \sum_{n=1}^{N} e(f(n)) \right|^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} e(f(m) - f(n))$$
$$= N + 2\mathrm{Re} \sum_{\ell=1}^{N-1} \sum_{n=1}^{N-\ell} e(f(n+\ell) - f(n)).$$

Note that $f(n+\ell) - f(n) = g_\ell(n)$ is a polynomial of degree $k-1$.

This process is known as **Weyl differencing**.

One can continue the process $k-1$ times and reduce the exponent to a linear polynomial.

In $\mathbb{R}$, a sequence $(a_n)_{n=1}^{\infty}$ of real numbers is **equidistributed** (mod 1) if for any interval $I \subset [0, 1)$, we have

$$\lim_{N \to \infty} \frac{\#\{a_n : 1 \leq n \leq N \text{ and } \{a_n\} \in I\}}{N} = |I|,$$

where $\{a\}$ is the fractional part of a.

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Theorem (Weyl, 1916)

If $f(x)$ is a polynomial with real coefficients and at least one of the non-constant coefficients is irrational, then the sequence $\{f(n)\}$ is equidistributed (mod 1).

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In the same paper, using the idea of differencing, Weyl also proved the famous inequality(Weyl's ineq), although it was given in a less explicit form.

Theorem (Weyl's inequality, an explicit form)

Suppose that $f(x) = \sum_{j=0}^{k} \alpha_j x^j \in \mathbb{R}[x]$, *and that* $|\alpha_k - a/q| < q^{-2}$, $(a,q) = 1$. *Then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{N} e(f(n)) \ll_{k,\varepsilon} N^{1+\varepsilon} \left(\frac{1}{q} + \frac{1}{N} + \frac{q}{N^k} \right)^{2^{1-k}} .$$

Theorem (Weyl's inequality, an inverse form)

Given $0 < \eta \le 2^{1-k}$, *for any* $\varepsilon > 0$, *if* N *is sufficiently large in terms of* ϵ *and* η, *and*

$$\left| \sum_{n=1}^{N} e(f(n)) \right| > N^{1-\eta},$$

then there are $(a,q) = 1$, *such that*

$$q < Z_{\eta,\varepsilon,k} = N^{\varepsilon + 2^{k-1}\eta} \quad \text{and} \quad |q\alpha_k - a| < Z_{\eta,\varepsilon,k}/N^k.$$

Weyl's inequality over primes in $\mathbb{Z}$

Theorem (Harman)

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$$\sum_{p \le N} (\log p) e(f(p)) \ll_{k,\varepsilon} N^{1+\varepsilon} \left(\frac{1}{q} + \frac{1}{N^{1/2}} + \frac{q}{N^k} \right)^{4^{1-k}} .$$

As a key ingredient in the Hardy-Littlewood Method, the Weyl-type inequality is applied in many problems.

- Waring's problem, Goldbach's problem...
- Diophantine inequalities, Diophantine equations...
- Sumsets problems, Sequences...
- Riemann zeta-function, L-functions...

Ring of polynomials over $\mathbb{F}_q$

Let $\mathbb{F}_q[t]$ be the polynomial ring over a finite field with q elements and characteristic p.

Let

$$\mathbb{K} = \mathbb{F}_q(t) = \left\{ \frac{x}{y} : x, y \in \mathbb{F}_q[t], y \neq 0 \right\}$$

be the field of fractions, and let

$$\mathbb{K}_\infty = \mathbb{F}_q((1/t)) = \left\{ \sum_{j=-\infty}^{N} a_j t^j : a_j \in \mathbb{F}_q, N \in \mathbb{Z} \right\}.$$

For $\alpha = \sum_{j=-\infty}^{N} a_j t^j \in \mathbb{K}_\infty$ with $a_N \neq 0$, we define $\mathrm{ord}(\alpha) = N$ and $|\alpha| = q^{\mathrm{ord}\,\alpha}$. In particular, $\mathrm{ord}(0) = -\infty$.

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Here, $\mathbb{F}_q[t]$, $\mathbb{K}$, $\mathbb{K}_\infty$ play the roles of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

Exponential function on $\mathbb{K}_\infty$

Define $\{\alpha\} = \sum_{j=-\infty}^{-1} a_j t^j$ to be the **fractional part** of α and let $\mathrm{res}(\alpha) = a_{-1}$. Then,

$$\mathbb{T} = \left\{ \sum_{j=-\infty}^{-1} a_j t^j : a_j \in \mathbb{F}_q \right\}$$

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Let $\mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p$ denote the trace map. Then for $\alpha \in \mathbb{K}_\infty$, the exponential function is defined as

$$e(\alpha) := e^{2\pi i \cdot \mathrm{tr}(\mathrm{res}\alpha)/p}.$$

This is an additive character on $\mathbb{K}_\infty$ and analogous to $e^{2\pi i x}$ in $\mathbb{R}$. We can use this function to study additive problems in function fields.

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- If $k < p = \mathrm{char}(\mathbb{F}_q)$, then one can repeat Weyl differencing and prove analogous results.
- If $k \geq p$, Weyl differencing is problematic. Look at the leading coefficient of $f(x)$. If we do $f(x+h) - f(x)$, $k-1$ times, we end up having a factor of $k!$ in the final leading coefficient, which is 0 when $k \geq p$.

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Y.-R. Liu and T. Wooley (2010), in their *Waring's problem* paper, overcame the barrier of $k < p$ in function fields, by using large sieve and Vinogradov's mean value theorem (VMVT).

Carlitz's Example

For any $x = \sum_{j=0}^{n} c_j t^j \in \mathbb{F}_q[t]$, we have $x^p = \sum_{j=0}^{n} c_j^p t^{jp} \in \mathbb{F}_q[t^p]$.

Example.(Carlitz, 1952) Let

$$\mathcal{C} = \left\{ \alpha : \alpha = \sum_{i=-\infty}^{n} c_i t^i, c_{-jp-1} = 0 \text{ for all } j \right\},$$

so that $e(\alpha x^p) = 1$ for all $x \in \mathbb{F}_q[t]$.

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Weyl-type inequality: if $|\sum e(\alpha x^p)|$ is large, can the leading coefficient α be well-approximated by rationals with small denominators?
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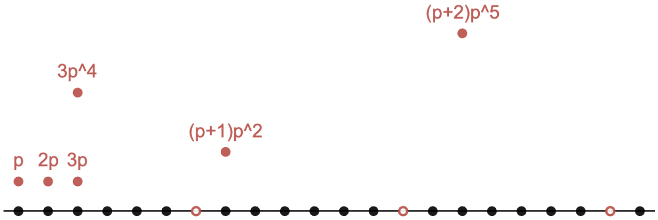
Example. For polynomials like $f(x) = \alpha x^p + \beta x$, it is not possible to determine the Diophantine approximation of α or β by the Weyl sum, since x^p and x interfere with one another.

Q: Given $f(x) = \sum_{j \in \mathcal{K}} \alpha_j x^j \in \mathbb{K}_\infty[x]$ supported on $\mathcal{K} \subset \mathbb{Z}^+$, which coefficients satisfy Weyl-type inequalities?

Example

Suppose $p = 7$ and $\mathcal{K} = ([1, 3p+1] \cap \mathbb{Z}) \cup \{p^3 + p^2, 3p^4, p^6 + 2p^5\}$.

To visualize it, we plot $\mathcal{K}$ on the number line in the following way.



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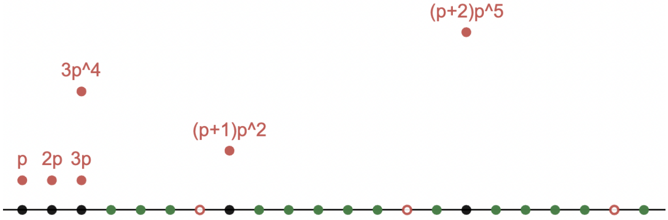

Ideally, the set of indices (in green) without interference is the largest subset of $\mathcal{K}$ on which Weyl's inequality applies.

Given a finite set $\mathcal{K} \subset \mathbb{Z}^+$, define the set (without interference)

$$\mathcal{I}_{\mathcal{K}} = \{k \in \mathcal{K} : p \nmid k, kp^v \notin \mathcal{K} \text{ for any positive integer } v\}.$$

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1. Define the **shadow** of $\mathcal{K}$ to be $\mathcal{S}(\mathcal{K}) := \{j \in \mathbb{Z}^+ : p \nmid \binom{r}{j} \text{ for some } r \in \mathcal{K}\}$.
2. Define $\mathcal{K}^* := \{k \in \mathcal{K} : p \nmid k \text{ and } p^v k \notin \mathcal{S}(\mathcal{K}) \text{ for any } v \in \mathbb{Z}^+\}$ to "remove" interfering coefficients (indices) on the shadow.
3. For $\mathcal{K}_0 = \mathcal{K}$, $\mathcal{K}_n = \mathcal{K}_{n-1} \setminus \mathcal{K}^*_{n-1}$, we define $\widetilde{\mathcal{K}} := \bigcup_{n \geq 0} \mathcal{K}^*_n$.

Lê-Liu-Wooley proved a Weyl-type inequality for all coefficients α_j with $j \in \widetilde{\mathcal{K}}$.

Note that

$$\widetilde{\mathcal{K}} \subset \mathcal{I}_{\mathcal{K}} \subset (\mathcal{K} \setminus p\mathbb{Z}).$$

Theorem (Lê-Liu-Wooley, 2023)

Fix q and a finite set $\mathcal{K} \subset \mathbb{Z}^+$. There exist positive constant c and C depending only on $\mathcal{K}$ and q, such that following holds. Let $\epsilon > 0$ and N sufficiently large (in terms of $\mathcal{K}, \epsilon, q$). Let $f(x) = \sum_{r \in \mathcal{K}} \alpha_r x^r \in \mathbb{K}_\infty[x]$. If

$$\left| \sum_{\deg x < N} e(f(x)) \right| \geq q^{N-\eta},$$

for some $\eta \in (0, cN]$. Then for each $k \in \widetilde{\mathcal{K}}$ there exist $a \in \mathbb{F}_q[t]$ and monic $g \in \mathbb{F}_q[t]$ such that

$$|g\alpha_k - a| < \frac{q^{\epsilon N + C\eta}}{q^{kN}} \quad \text{and} \quad |g| \leq q^{\epsilon N + C\eta}.$$

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- $f(x) = \alpha_k x^k + \cdots$ with $(k, p) = 1$.
- $f(x) = \alpha_\ell x^\ell + \cdots + \alpha_k x^k + \cdots$, with $(k, p) = 1$ and $k > \ell/p$.
- $f(x) = \sum_{1 \leq j \leq k, (j,p)=1} \alpha_j x^j$. In this case, $\widetilde{\mathcal{K}} = \mathcal{I} = \mathcal{K}$.

Define the von Mangoldt function over $\mathbb{F}_q[t]$ by $\Lambda(x) = \deg(P)$, if $x = cP^r$ for some monic irreducible P, zero otherwise.

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Theorem (Champagne-G.-Lê-Liu, 2023+)

Let $\mathcal{K} \subset \mathbb{Z}^+$ be a finite set and $k \in \mathcal{I}_{\mathcal{K}}$. There exist constants $c_k, C_k > 0$ (depending on $k, \mathcal{K}, q$) such that the following holds:
Let $\epsilon > 0$ and N be sufficiently large in terms of $\mathcal{K}$, ϵ and q. Suppose that $f(u) = \sum_{r \in \mathcal{K} \cup \{0\}} \alpha_r u^r \in \mathbb{K}_\infty[u]$ satisfying the bound

$$\left| \sum_{x \in \mathbb{A}_N} \Lambda(x) e(f(x)) \right| \geq q^{N-\eta},$$

for some η with $0 < \eta \leq c_k N$. Then, there exist $a_k \in \mathbb{F}_q[t]$ and monic $g_k \in \mathbb{F}_q[t]$ such that

$$|g_k \alpha_k - a_k| < \frac{q^{\epsilon N + C_k \eta}}{q^{kN}} \qquad \text{and} \qquad |g_k| \leq q^{\epsilon N + C_k \eta}.$$

Application 1: Equidistribution Theorem

Like Weyl proved the equidistribution theorem, Lê-Liu-Wooley (in the same paper) proved the next theorem.

Theorem (Lê-Liu-Wooley, 2023)

Let $f(u) = \sum_{r \in \mathcal{K} \cup \{0\}} \alpha_r u^r$ be a polynomial supported on $\mathcal{K} \subset \mathbb{Z}^+$ with coefficients in $\mathbb{K}_\infty$. Suppose α_k is irrational for some $k \in \widetilde{\mathcal{K}}$. Then the sequence $(f(x))_{x \in \mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$.

Remarks:

- **Carlitz** (1952) gave a family of irrational α that $e(\alpha x^p) = 1$ for all $x \in \mathbb{F}_q[t]$, thus equidistribution does not hold for $f(x) = \alpha x^p$.
- **Bergelson-Leibman** (2015) proved a similar equidistribution theorem independently using ergodic-theoretic methods.

$\mathbb{P} = \{x \in \mathbb{F}_q[t] : \text{monic irreducible}\}$.

Theorem (Champagne-G.-Lê-Liu, 2023+)

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Theorem (Champagne-G.-Lê-Liu, 2023+)

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- **Carlitz** (1952): the result may not hold for $f(x) = \alpha x^p$.
- **Rhin** (1972) proved the theorem when $\mathcal{K} = \{1\}$.
- *Difficulty:* The space $\mathbb{P}$ is not self-similar as $\mathbb{F}_q[t]$. A Weyl-type inequality does not immediately imply the equidistribution theorem.
    1. We prove for the special case $\widetilde{\mathcal{K}} = \mathcal{I}_\mathcal{K} = \mathcal{K}$, for which we further prove an epsilon-free version of Weyl's inequality.
    2. Then we prove the equidistribution theorem on $\mathcal{I}_\mathcal{K}$ for general $\mathcal{K}$, using Jérémy Champagne's argument.

Application 2: Additive inequality of irreducible powers

Let $\mathbb{P}_{kN}^k = \{x^k : x \text{ is monic irreducible}, \deg(x^k) = kN\}$.

Theorem (G.)

Suppose $(p,k) = 1$ and $k \geq 2$. Let N be a large number. Let $\mathcal{A}$ be a set of polynomials in $\mathbb{F}_q[t]$ of degree less than kN and $0 < \frac{|\mathcal{A}|}{q^{kN}} = \delta < e^{-2}$. Then we have

$$\frac{|\mathcal{A} + \mathbb{P}_{kN}^k|}{q^{kN}} > \delta^{\frac{4\log(2) + c_q \log(k)}{\log\log(1/\delta)}}$$

for some $c_q > 0$.

- It is different from the analog in $\mathbb{Z}$ that the theorem is not true when $p \mid k$.
- Among all monic degree-kN polynomials, the proportion (density) of $\mathbb{P}_{kN}^k$ is very tiny. However, $\mathcal{A} + \mathbb{P}_{kN}^k$ is significantly denser than $\mathcal{A}$ for every small density set $\mathcal{A}$.

Ingredients of the Proof

Ingredients of Lê-Liu-Wooley's original method include

- Weyl's shift,
- Large sieve inequality (**Hsu**),
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More tools for irreducible elements:

- Vaughan's identity in $\mathbb{F}_q[t]$.
- A bootstrap argument. (Iterate LLW's argument multiple times.)
- Major arc estimates for removing the epsilon.
- A nice self-duality property of $\mathbb{K}_\infty$.

To help sketch the arguments, we introduce the following notation:

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Moreover,

$$\mathbb{A}_N := \{x \in \mathbb{F}_q[t] : \text{monic } \deg(x) = N\}.$$

This is the analog of the dyadic interval $[N, 2N)$ in integers.

Sketch of Lê-Liu-Wooley's argument

Lemma (Weyl's shift)

Let $\mathcal{A} \subset \mathbb{F}_q[t]$ be a multiset consisting of elements of degree less than N. We have

$$\sum_{x \in \mathbb{A}_N} e(f(x)) = \#(\mathcal{A})^{-1} \sum_{x \in \mathbb{A}_N} \sum_{y \in \mathcal{A}} e(f(y+x))$$

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Proof. For each y with $\deg(y) < N$, we have

$$\sum_{x \in \mathbb{A}_N} e(f(x)) = \sum_{x \in \mathbb{A}_N} e(f(x + y)).$$

Summing $y \in \mathcal{A}$, the lemma follows.

- The choice of $\mathcal{A}$ is very flexible!
- Instead of looking at a sum over $\mathbb{A}_N$, we turn attention on summing $e(g_x(y)) = e(f(x + y))$ over $y \in \mathcal{A}$.
- The new polynomial $g_x(y)$ is supported on the **shadow**. (Bad)

Sketch of Lê-Liu-Wooley's argument

1. Based on Dirichlet's approximation, we take a multiset $\mathcal{A} = \{\ell u\}$ that "fit" the approximation and (Weyl) shift the sum onto $\mathcal{A}$.
   - This turns the original sum into a bilinear sum.
   - It creates well-spaced (leading) coefficients $\{\alpha \ell^k\}$, i.e. distinct elements are at least $q^{-\lambda}$ apart in $\mathbb{T}$ for some $\lambda > 0$ (depending on the Diophantine approximation of α).
2. Then, we apply Hölder's inequality and Hsu's large sieve inequality to convert the bilinear sum into Vinogradov's mean value problem.
3. Finally, we apply Liu-Wooley's VMVT. The final upper estimate depends on q^{λ} (and hence the Diophantine approximation of α).

Vaughan's identity

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Let $1 \leq U, V \leq N$. For every monic $x \in \mathbb{F}_q[t]$ with $\deg(x) < U$, we have

$$\Lambda(x) = a_1(x) + a_2(x) + a_3(x),$$

where

$$a_1(x) = -\sum_{\substack{uvw=x \\ u \in \mathbb{G}_U \\ v \in \mathbb{G}_V}} \Lambda(u)\mu(v), \qquad a_2(x) = \sum_{\substack{uv=x \\ u \in \mathbb{G}_V}} \deg(u)\mu(v),$$

$$a_3(x) = \sum_{\substack{uvw=x \\ \deg(u) \geq U \\ \deg(v) \geq V}} \Lambda(u)\mu(v),$$

and the sums are over monic polynomials.

By Vaughan's identity,

$$S(N, f) = \sum_{x \in \mathbb{A}_N} \Lambda(x) e(f(x)) = S_1 + S_2 + S_3.$$

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- **Type I sums**: $$J_1 = \sum_{u \in \mathbb{A}_L} \phi(u) \sum_{v \in \mathbb{A}_{N-L}} e(f(uv)).$$

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  S_1 and S_2 can be decomposed as linear combination of Type I sums. In particular, when $L = 0$, this is an ordinary exponential sum.

- **Type II sums**:

  $$J_2 = \sum_{u \in \mathbb{P}_L} \sum_{v \in \mathbb{G}_{N-L}} \psi(v) e(f(uv)),$$

  where $\mathbb{P}_L$ is the set of monic irreducible polynomials of degree L. Using triangle inequality, S_3 can be bounded by Type II sums.

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In our proof, we consider the problem for the bilinear sums.

- **Type I sums**

$$J_1 = \sum_{u \in \mathbb{A}_L} \phi(u) \sum_{v \in \mathbb{A}_{N-L}} e(f(uv)), \quad \text{for } 0 \leq L \leq N - 2M$$

- **Type II sums**

$$J_2 = \sum_{u \in \mathbb{P}_L} \sum_{v \in \mathbb{G}_{N-L}} \psi(v) e(f(uv)), \quad \text{for } 0 \leq L \leq N/2.$$

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The difficulty is the to obtain the same quality of the rational approximation of α_k simultaneously for all (large) L in the red range.

Estimate of Type II Sums

Consider

$$J_2 = \sum_{u \in \mathbb{P}_L} \sum_{v \in \mathbb{G}_{N-L}} \psi(v) e(f(uv)).$$

- One can partition $\mathbb{P}_L = \cup_i \mathcal{A}_i$ (Very flexible)
- After triangle inequality, to study J_2, it suffices to look at the sum over $\mathcal{A}$:

$$\sum_{u \in \mathcal{A} \subset \mathbb{P}_L} \sum_{v \in \mathbb{G}_{N-L}} \psi(v) e(f(uv)).$$

These two bullet points are parallel to Weyl's shift.

- We begin with Dirichlet's theorem. Accordingly, we pick a family of sets $\mathcal{A}$ that "fit" the trivial approximation:

$$|J_2| \leq \sum_i \left| \sum_{u \in \mathcal{A}_i} \sum_{v \in \mathbb{G}_{N-L}} \psi(v) e(f(uv)) \right|.$$

- After Holder's inequality, Hsu's large sieve, and Liu-Wooley's theorem, we end up having

  If $|J_2| > Tq^{N-M}$ where $|\psi| \leq T$, then there are $(a, b) = 1$ with

$$|b\alpha - a| < q^{-kN+L}, \quad |b| < q^{M}. \tag{1}$$

The approximation (1) is worse than what we want when $L > M$, but this is still much better than the trivial approximation.

Remark. The process in the second bullet point is independent of what $\mathcal{A}$ is.

Bootstrap the quality of the approximation

$$|J_2| \leq \sum_i \left| \sum_{u \in \mathcal{A}_i} \sum_{v \in \mathbb{G}_{N-L}} \psi(v) e(f(uv)) \right|$$

Next, we repeat LLW's argument again.

- Suppose $|J_2| > Tq^{N-M}$. Then we have approximation (1) in hand, which is much better than the trivial approximation.
- Next, we find a new family of $\mathcal{A}$s that "fit" the approximation (1). We are going to do LLW's process over this new family of $\mathcal{A}$.
- After Holder's inequality, Hsu's large sieve, and Liu-Wooley's theorem, we end up having:

  If $|J_2| > Tq^{N-M}$ then there are $(a, b) = 1$ with

$$|b\alpha - a| < q^{-kN+M}, \quad |b| < q^M. \tag{2}$$

Further remarks

- For $J_2 = \sum_{u \in \mathbb{P}_L} \sum_{v \in \mathbb{G}_{N-L}} \psi(v) e(f(uv))$, we can do $M \leq L \leq N/2$ at this moment.

  The barrier $N/2$ can be relaxed to N if one applies Vaughan's identity to the bilinear sum and repeats the whole process again.

- In the classical Vaughan/Vinogradov's Type I/II method, type II is usually the more difficult one, but in our case, Type II is the easier one.

Generalizing $\tilde{\mathcal{K}}$ to $\mathcal{I}$

Lemma (Self-duality)

For any $v \in \mathbb{Z}^+ \cup \{0\}$ and $\alpha \in \mathbb{K}_\infty$, there exists $\tau = \tau_v(\alpha) \in \mathbb{K}_\infty$ such that

$$e(\alpha x^{rp^v}) = e(\alpha (x^r)^{p^v}) = e(\tau x^r)$$

Given a finite $\mathcal{K} \subset \mathbb{Z}^+$, $\mathcal{R} = \mathcal{R}_\mathcal{K} = \{r : p \nmid r, rp^v \in \mathcal{K} \text{ for some integer } v\}$.

Using the above lemma, we can simplify the sum as

$$\sum_x e\Big(\sum_{j \in \mathcal{K}} \alpha_j x^j\Big) = \sum_x e\Big(\sum_{j \in \mathcal{R}} \tau_j x^j\Big).$$

Note that $\mathcal{I} \subseteq \mathcal{K} \cap \mathcal{R}$ and $\alpha_j = \tau_j$ when $j \in \mathcal{I}$.

We know how to estimate the sum over $\mathcal{R}$ by LLW, since $\tilde{\mathcal{R}} = \mathcal{R}$.

Thank You !