Vertex-transitive graphs with large automorphism groups

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Definitions

All graphs are finite, connected and simple (undirected, loopless, no multiple edges).

An arc in a graph is an ordered pair of adjacent vertices.

A graph Γ is G-vertex-transitive (G-arc-transitive) if $G \leq \mathrm{Aut}(\Gamma)$ acts transitively on the the vertex-set (arc-set) of Γ.

Arc-transitivity implies edge-transitivity and vertex-transitivity (under mild hypothesis).

Tutte's Theorem and consequences

Theorem (Tutte 1947)

If Γ is a 3-valent arc-transitive graph and v is a vertex of Γ, then $|\mathrm{Aut}(\Gamma)_v| \leq 48$.

By Orbit-Stabiliser, $|\mathrm{Aut}(\Gamma)|$ grows linearly with respect to $|\mathrm{V}\Gamma|$.

Allows one (e.g. Conder) to enumerate these graphs up to "large" order (say 10000).

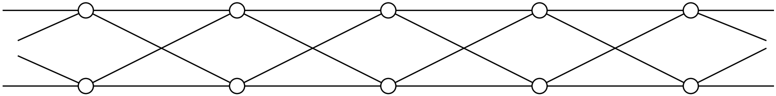
Theorem (Potočnik, Spiga, V 2017)

The number of 3-valent arc-transitive graphs of order at most n is

$$\sim n^{c \log n}$$

Valency four

Let $\mathrm{PX}(r,1) = \mathrm{C}_r[\overline{K}_2]$:



$\mathrm{PX}(r,1)$ is connected, arc-transitive and 4-valent. Moreover $|\mathrm{Aut}(\mathrm{PX}(r,1))| \geq 2^r$ which is exponential in $|\mathrm{VPX}(r,1))| = 2r$.

More generally, we can define $\mathrm{PX}(r,s)$. We have $|\mathrm{VPX}(r,s))| = 2^s r$ and $|\mathrm{Aut}(\mathrm{PX}(r,s))| \geq 2^{r-s}$.

Similar examples for 3-valent vertex-transitive graphs.

Valency four

Theorem (Potočnik, Spiga, V 2015)

There exists $c \in \mathbb{R}$ such that, if Γ is a 4-valent arc-transitive graph, then either

1. $|\mathrm{Aut}(\Gamma)_v| \leq c|\mathrm{V}\Gamma|$, *or*
2. $\Gamma \cong \mathrm{PX}(r, s)$ *for some r, s.*

In first case, $|\mathrm{Aut}(\Gamma)|$ still grows slowly (polynomially) with $|\mathrm{V}\Gamma|$.

Many consequences of Tutte's Theorem still go through.

Similar results about 3-valent vertex-transitive graphs.

Local action

Let Γ be a G-vertex-transitive graph and let v be a vertex of Γ.

$G_v^{\Gamma(v)}$ denotes the permutation group induced by the action of G_v on the neighbourhood $\Gamma(v)$.

(Γ, G) is locally-L if $G_v^{\Gamma(v)}$ is permutation isomorphic to L.

Example

$(\mathrm{PX}(r, 1), \mathrm{Aut}(\mathrm{PX}(r, 1)))$ is locally-D_4.

Types of growth

We can (informally) define the "growth rate" of a permutation group L: in the class of locally-L pairs (Γ, G), how fast can $|G_v|$ grow as a function of $|\mathrm{V}\Gamma|$? (Best upper bound)

The fastest possible growth is exponential (for example D_4).

Tutte's Theorem: D_3 has "constant growth".

All transitive groups of degree 4 except D_4 have constant growth.

D_n has constant growth when n is odd, exponential growth when $n = 4$, and (non-constant) polynomial growth otherwise.

Problem

Find the growth rate of every permutation group.

What's known

Weiss, Trofimov: Transitive groups of prime degree have constant growth.

There are 37 transitive groups of degree at most 7.

Theorem (Many people, last case in 2021)

- 26 *have constant growth.*
- 10 *have exponential growth.*
- 1 (D_6) *has polynomial growth.*

50 transitive groups of degree 8, growth known for all but 3.

Example: $\mathrm{Sym}(4)$ transitive on 6 points

We have

$$L := S_4 \cong \mathbb{Z}_2^2 \rtimes S_3 < \mathbb{Z}_2^3 \rtimes S_3 = \mathbb{Z}_2 \wr S_3 \leq S_6,$$

where $\mathbb{Z}_2^2$ consists of the codimension 1-subspace of elements with entries having sum 0 in $\mathbb{Z}_2$.

We want to construct locally-L pairs with size of groups exponential in the order of the graph.

We start with a locally-$\mathrm{Sym}(3)$ 3-valent graph, then take the lexicographic product with an edgeless graph of order 2.

We need to find a "large" subgroup of the automorphism group that is locally-L.

The eigenspaces over $\mathbb{Z}_2$ of the 3-valent graph turns out to be relevant!

$\mathrm{Sym}(4)$ transitive on 6 points, continued

We need an infinite family of 2-arc-transitive 3-valent graphs with the property that the dimension of their eigenspaces over $\mathbb{Z}_2$ grows linearly with the order of the graphs.

Easy to find a family of candidates using the census, very nice description, but hard to prove the lower bound on the size of the eigenspace.

More results of this type?

The three remaining groups of degree 8 are of this form.

Questions about growth

Are there groups of intermediate growth?

Conjecture (Potočnik, Spiga, V 2011)

Constant growth $\iff$ *semiprimitive*

(A permutation group is semiprimitive if every normal subgroup is either transitive or semiregular.)

(Conjecturally) Characterize groups of polynomial growth.

Growth too coarse?

Two related issues:

1. When proving groups have exponential growth, we usually use graphs which obviously have exponential growth and then try to find appropriate subgroups.
2. Even if group has exponential growth, the exponential examples may be rare/well understood (such as for D_4).

Goal is to find analogues to the result for 4-valent arc-transitive graphs: either we have polynomial growth or we know the graphs.

Property $(\star)$

Definition

A permutation group has property $(\star)$ if it has an abelian normal subgroup which is not semiregular. (Point-stabilisers are non-trivial.)

Theorem (Praeger, Xu 1989)

Let Γ be a 4-valent G-arc-transitive graph. If G has property $(\star)$, then $\Gamma \cong \mathrm{PX}(r, s)$ for some r, s.

Question

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If Γ is a G-arc-transitive graph of valency k and G_v is soluble, then either

1. *G has property $(\star)$ or*
2. *$|G|$ is at most polynomial (depending on k) in $|V\Gamma|$.*

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Conjecture

(Informal) Let Γ be a G-arc-transitive graph of valency k. If N is an abelian normal subgroup of G and G/N has property $(\star)$, then either

1. *G has property $(\star)$,*
2. *$|G_v|$ is at most polynomial (depending on k) in $|V\Gamma|$.*

Other characterisations of PX graphs

Theorem (Potočnik, Spiga 2021)

Let Γ be a 4-valent arc-transitive graph. Either

1. $\Gamma \cong \mathrm{PX}(r, s)$ *for some* r, s,
2. *No non-identity automorphism of* Γ *fixes more than* $1/3$ *of the vertices,*
3. *Finitely many exceptions.*

Need to generalise PX graphs

Similar results for

- 3-valent vertex-transitive graphs (Potočnik, Spiga V 2015)
- locally-$\mathbb{Z}_p \wr \mathbb{Z}_2$ graphs (Spiga, V 2016)
- 2-valent arc-transitive digraphs (Potočnik, Spiga V 2015)

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All use results about (di)graphs closely related to PX graphs (global "cyclic" structure).

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Thank you!