Moments of higher derivatives related to Dirichlet L-functions

Samprit Ghosh

UNIVERSITY OF CALGARY

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Introduction

- The distribution of values of Dirichlet L-functions at $s = 1$, i.e., $L(1, \chi)$ for variable χ has a vast literature, whereas the study of the same for the logarithmic derivative $\frac{L'}{L}(1, \chi)$ is fairly recent!

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- In 2009, Ihara, Murty and Shimura computed "(a,b)-th moment" i.e.

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}\left(\frac{L'(1,\chi)}{L(1,\chi)}\right) \quad \text{where } P^{(a,b)}(z)=z^a\overline{z}^b,$$

  m is a prime number and X_m is the set of all non-principal Dirichlet characters.

- In today's talk we will present generalization of their results to higher derivatives, i.e. we'll look at

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}\left(\mathscr{L}^{(r)}(1,\chi)\right) \quad \text{where } \mathscr{L}^{(r)}(s,\chi)=\frac{d^r}{ds^r}\frac{L'(s,\chi)}{L(s,\chi)}$$

Preliminaries

The Riemann zeta function

$$\zeta(s) = \sum_{n} \frac{1}{n^s} \quad \text{for } \mathrm{Re}(s) > 1$$

has a simple pole at $s = 1$ and can be analytically continued everywhere else in the complex plane.

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$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1).$$

Where γ is the famous Euler-Mascheroni constant defined as

$$\gamma := \lim_{x \to \infty} \left(\sum_{n \le x} \frac{1}{n} - \log x \right).$$

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Motivated by this, Ihara introduced a generalization of γ for any number field K. The Dedekind zeta function of K is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s}$$

for $\mathrm{Re}(s) > 1$, where the sum is taken over all integral ideals $\mathfrak{a}$ of $\mathcal{O}_K$.

Euler-Kronecker constants

If the Laurent series of $\zeta_K(s)$ at $s = 1$ is given by

$$\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1),$$

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It turns out γ_K is the constant term in the Laurent series of the logarithmic derivative of $\zeta_K(s)$ at $s = 1$.

$$\frac{\zeta_K'}{\zeta_K}(s) = \frac{-1}{s-1} + \gamma_K + O(s-1).$$

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Definition. If we write the Laurent series

$$\frac{\zeta_K'(s)}{\zeta_K(s)} = \frac{-1}{s-1} + \gamma_{K,0} + \gamma_{K,1}(s-1) + \gamma_{K,2}(s-1)^2 + \cdots \tag{1}$$

We will call $\gamma_{K,n}$ as the $\mathbf{n^{th}}$ **Euler-Kronecker constant.**

Motivation

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- Consider the completed zeta function

$$\xi_K(s) = s(s-1)2^{r_2}\left(\frac{\sqrt{|d_K|}}{2^{r_2}\pi^{n/2}}\right)^s \Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\zeta_K(s)$$

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Li's Criterion (Xian-Jin Li, 1997)

Consider the sequence $(n \geq 1)$

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}[s^{n-1}\log \xi_K(s)]\Big|_{s=1}$$

Then Riemann Hypothesis holds $\Leftrightarrow$ $\lambda_n \geq 0$ for all n.

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Then Riemann Hypothesis holds $\Leftrightarrow$ $\lambda_n \geq 0$ for all n.

- Brown, F. (2004) showed an effective version of this theorem, relating non-negativity of the first m terms of the sequence to zero free regions around $s = 1$.

Motivation

In particular, a corollary of Brown's result :

$\lambda_2 \geq 0 \ \Rightarrow$ Non-existence of the exceptional Siegel zero.

Note : A well-known result of Stark says that for $0 < c < \frac{1}{4}$, $\zeta_K(s)$ *has at most one zero in the region*

$$1 - \frac{c}{\log d_K} \leq \sigma \leq 1, \ \ |t| \leq \frac{c}{\log d_K}$$

where $s = \sigma + it$. *This zero, if it exists, is necessarily real and simple. We call this an exceptional Siegel zero.*

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– Note that λ_2 corresponds to the first Euler-Kronecker constant $\gamma_{K,1}$.
– This motivated us to study $\gamma_{K,1}$, we then realized many of the results/techniques generalize to higher Euler-Kronecker constants as well.

Work of Ihara, Murty and Shimura

- Let K be a number field and χ be a primitive Dirichlet character on K. Let $L(s,\chi)$ be the L-function associated to it. Ihara et al., studied $\frac{L'}{L}(1,\chi)$.

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Ihara, Murty and Shimura (2009)

If $\chi \neq \chi_0$, then

$$\frac{L'}{L}(1,\chi) = -\lim_{x\to\infty} \Phi_{K,\chi}(x) \tag{2}$$

where

$$\Phi_{K,\chi}(x) = \frac{1}{x-1} \sum_{N(P)^k \leq x} \left(\frac{x}{N(P)^k} - 1 \right) \chi(P)^k \log N(P) \quad (\text{ for } x > 1)$$

Here, k is a positive integer and the sum is taken over non-archimedean primes. Under GRH,

$$\left| \frac{L'}{L}(1,\chi) \right| < 2 \log\log \sqrt{d_\chi} + 1 - \gamma_{K,0} + O\left(\frac{\log|d_K| + \log\log d_\chi}{\log d_\chi} \right)$$

Here, $d_\chi = |d_K| N(\mathfrak{f}_\chi)$ and $\gamma_{K,0}$ is the Euler-Kronecker constant of K.

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Theorem (G.)

For $\chi \neq \chi_0$, we have, unconditionally

$$\mathscr{L}^{(n)}(1,\chi) = \lim_{x\to\infty} (-1)^{n+1}\, \Psi_K(\chi,n,x)$$

where

$$\Psi_K(\chi,n,x) = \frac{1}{x-1}\sum_{N(P)^k\leq x} k^n \left(\frac{x}{N(P)^k} - 1\right)\chi(P)^k\,(\log N(P))^{n+1} \quad (\text{ for } x>1)$$

Here, k is a positive integer and the sum is taken over non-archimedean primes.
Under GRH,

$$\mathscr{L}^{(n)}(1,\chi) \ll \frac{2^n}{n!}(\log(n!) + 2\log\log\sqrt{d_\chi} - \gamma_{K,0})(\log(n!) + \log\log\sqrt{d_\chi})^n$$

Here, $d_\chi = |d_K|N(\mathfrak{f}_\chi)$ and $\gamma_{K,0}$ is the Euler-Kronecker constant of K.

Main idea for the proof

In two different ways we evaluate the integral :

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-\mu}}{s-\mu}\mathscr{L}^{(n)}(\chi,s)\,ds \qquad \text{for } c \gg 0$$

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On one hand we take the logarithmic derivative of the Euler product of $L(s,\chi)$

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On the other, taking the logarithmic derivative of the Hadamard product of the completed L-function :

$$\mathscr{L}(s,\chi)=C-\frac{a}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)-\frac{a'}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right)-r_2\frac{\Gamma'}{\Gamma}(s)+\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) \tag{4}$$

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- Lots of residue computation!

Moments : Some history

Let m be a prime and X_m denote the set of all non-principal multiplicative characters $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ and $L(s, \chi)$ denote the corresponding Dirichlet L-function.

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Theorem. (Payley, Selberg 1931)

$$\frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(1,1)}(L(1,\chi)) = \zeta(2) + O((\log m)^2/m)$$

This was later improved and by many authors. W. Zhang in 1990 generalized to the case of $P^{(k,k)}$.

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Theorem. (Ihara, Murty, Shimura 1931)

Unconditionally,

$$\frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(a,b)}(\mathscr{L}(1,\chi)) = (-1)^{a+b}\mu^{(a,b)} + O(m^{\varepsilon-1}) \quad (5)$$

for any $\varepsilon > 0$.

Moments : Some history

Here $\mu^{a,b}$ is a non-negative real number defined as follows :

$$\mu^{(a,b)} = \sum_{n=1}^{\infty} \frac{\Lambda_a(n)\Lambda_b(n)}{n^2} \quad \text{where} \quad \Lambda_k(n) = \sum_{n=n_1 \cdots n_k} \Lambda(n_1)\cdots\Lambda(n_k)$$

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- We wish to generalize their result for higher derivatives!

Moments of higher derivatives

• I was not able to find a good reference that studies moments of higher derivatives of $L(s,\chi)$ at $s=1$ but the case of $s=\frac{1}{2}$ has been studied by Soundararajan, Sono etc. For example, here is a recent result :

Theorem. (Sono, 2014)

For $k\geq 2$, $m\in\mathbb{Z}_{\geq 0}$ and for any $\epsilon>0$, under GRH, we have

$$\frac{1}{\phi(q)}\sideset{}{'}\sum_{\chi \pmod q} P^{(k,k)}\left(L^{(m)}\left(\frac{1}{2},\chi\right)\right)\ll(\log q)^{k^2+2km+\epsilon}$$

where $\sum'$ is over all primitive Dirichlet characters modulo q.

Moments of higher derivatives

Theorem (G.)

For any $\epsilon > 0$, we have, unconditionally,

$$\frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(a,b)}(\mathscr{L}'(1,\chi)) = (-1)^{a+b} \tilde{\mu}^{(a,b)} + O\left(m^{\epsilon - 1}\right)$$

where the implicit constant depends only on a, b.

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where the implicit constant depends only on a, b. If we define :

$$\ell^1 \Lambda_k(n) = \sum_{n=n_1 \cdots n_k} \Lambda(n_1) \cdots \Lambda(n_k)(\log n_1) \cdots (\log n_k) \quad \text{for } k > 0. \tag{6}$$

and for $k = 0$ we define it to be 1 if $n = 1$ and 0 otherwise.

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and for $k = 0$ we define it to be 1 if $n = 1$ and 0 otherwise.

For each pair (a, b) of non-negative integers, we define

$$\tilde{\mu}^{(a,b)} = \tilde{\mu}^{(b,a)} = \sum_{n=1}^{\infty} \frac{\ell^1 \Lambda_a(n)\, \ell^1 \Lambda_b(n)}{n^2} \tag{7}$$

Some notes on the theorem

- For example, note that $\tilde{\mu}^{(0,0)} = 1$, $\tilde{\mu}^{(a,0)} = 0$ for all $a > 0$, and in all other cases $\tilde{\mu}^{(a,b)} > 0$. In particular,

$$\tilde{\mu}^{(1,1)} = \sum_{n=1}^{\infty} \left(\frac{\Lambda(n) \log(n)}{n} \right)^2$$

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- **Further generalizations :** In fact, if we define for $k > 0$, $r \geq 0$

$$\ell^r \Lambda_k(n) = \sum_{n_1 n_2 \cdots n_k = n} \left(\prod_{i=1}^{k} \Lambda(n_i)(\log n_i)^r \right)$$

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Generalizing the theorem on moments

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For any $\epsilon > 0$, we have, unconditionally,

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where the implicit constant depends only on a, b.

Under GRH, the error term is

$$O\left(\frac{(\log m)^{(r+1)(a+b)+2}}{m}\right)$$

Outline of the proof

- Recall the function $\Psi(\chi, 1, x)$ related to $\mathscr{L}'(1, \chi)$:

$$\Psi(\chi, 1, x) = \frac{1}{x-1} \sum_{k, p^k < x} k\left(\frac{x}{p^k} - 1\right)\chi(p)^k (\log p)^2$$

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$$= \frac{1}{x-1} \sum_{n \leq x} \left(\frac{x}{n} - 1 \right) \chi(n)\Lambda(n)(\log n)$$

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One can then show an intermediate result : For each pair (a,b) of non-negative integers and for $x \geq m$, we have

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}\left(\Psi(\chi,1,x)\right) = \tilde{\mu}^{(a,b)} + O_{a,b}\left(\frac{(\log x)^{2(a+b+1)}}{m}\right) \qquad (9)$$

Proof Contd..

Let us write $X_m^\star = X_m \cup \{\chi_0\}$ and

$$\sum_{\chi \in X_m^\star} P^{(a,b)}\left(\Psi(\chi,1,x)\right) = \sum_{\chi \in X_m^\star} \Psi(\chi,1,x)^a \, \Psi(\overline{\chi},1,x)^b$$

Proof Contd..

Let us write $X_m^\star = X_m \cup \{\chi_0\}$ and

$$\sum_{\chi \in X_m^\star} P^{(a,b)}\left(\Psi(\chi,1,x)\right) = \sum_{\chi \in X_m^\star} \Psi(\chi,1,x)^a \, \Psi(\overline{\chi},1,x)^b$$

Lemma. For some $x > 1$, and $\chi \in X_m^\star$ if $g_\chi(x) = \sum_{n \le x} g(x,n)\chi(n)$ then,

$$\frac{1}{|X_m^\star|} \sum_{\chi \in X_m^\star} g_\chi(x)^a g_{\overline{\chi}}(x)^b = \sum_{j=1}^{m-1} \lambda^{(a)}(j,x)\, \lambda^{(b)}(j,x) \tag{10}$$

where

$$\lambda^{(k)}(j,x) = \sum_{\substack{n_1,\cdots,n_k < x \\ n_1 \cdots n_k \equiv j \pmod m}} \prod_{i=1}^{k} g(x,n_i)$$

for $k \ge 1$, and for $k = 0$ define $\lambda^{(0)}(j,x) = 1$ for $j = 1$ and 0 for $j > 1$. (Recall m here is a prime number and a, b non-negative integers.)

Proof Contd..

Now choosing $g(x,n) = \frac{1}{x-1}\left(\frac{x}{n} - 1\right)\Lambda(n)\log n$, we get, $g_\chi(x) = \Psi(\chi, 1, x)$.

Proof Contd..

Now choosing $g(x,n) = \frac{1}{x-1}\left(\frac{x}{n}-1\right)\Lambda(n)\log n$, we get, $g_\chi(x) = \Psi(\chi, 1, x)$. Then one can show

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- Once we have

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}\left(\Psi(\chi,1,x)\right) = \tilde{\mu}^{(a,b)} + O_{a,b}\left(\frac{(\log x)^{2(a+b+1)}}{m}\right)$$

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$$\lambda^{(k)}(j,x) = \frac{1}{(x-1)^k} \sum_{\substack{n_1,\cdots,n_k<x \\ n_1\cdots n_k \equiv j \ (\mathrm{mod}\ m)}} \prod_{i=1}^{k} \left(\frac{x}{n_i}-1\right)\Lambda(n_i)\log n_i$$

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to get the error term we then estimate

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} \left|P^{(a,b)}(\mathscr{L}'(1,\chi)) - P^{(a,b)}(\Psi(\chi,1,x))\right|$$

Proof Contd.

To start with, one uses an elementary inequality

$$|P^{(a,b)}(z+w)-P^{(a,b)}(z)| \leq (a+b)|w|(|z|+|w|)^{a+b-1}.$$

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Thus,

$$\left|P^{(a,b)}(\mathscr{L}'(1,\chi)) - P^{(a,b)}(\Psi(\chi,1,x))\right| \leq (a+b)\left|\Psi(\chi,1,x) - \mathscr{L}'(1,\chi)\right| \cdot \left(\left|\Psi(\chi,1,x) - \mathscr{L}'(1,\chi)\right| + \left|\mathscr{L}'(1,\chi)\right|\right)^{a+b-1}$$

We then use zero sum estimates to estimate these terms and get the result.

A Theorem on distribution

Theorem. (G.)

For any $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > 1$ there exists a function $M_\sigma : \mathbb{C} \to \mathbb{R}$ satisfying, $M_\sigma(w) \geq 0$, and $\int_{\mathbb{C}} M_\sigma(w)\, |dw| = 1$, such that

$$\mathrm{Avg}_\chi\, \Phi(\mathscr{L}'(\chi, s)) = \int_{\mathbb{C}} M_\sigma(w)\, \Phi(w)\, |dw| \tag{11}$$

holds for any continuous function Φ on $\mathbb{C}$. Here, if we write, $w = x + iy$ then $|dw| = (2\pi)^{-1} dxdy$.

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Here K be either $\mathbb{Q}$ or an imaginary quadratic number field, χ runs over all Dirichlet characters on K with prime conductors normalized by the condition $\chi(\wp_\infty) = 1$; The average of a complex valued function $\phi(\chi)$, is defined as :
$\mathrm{Avg}_\chi \phi(\chi) = \lim_{m\to\infty} \mathrm{Avg}_{N(\mathbf{f})\leq m} \phi(\chi)$ where

$$\mathrm{Avg}_{N(\mathbf{f})\leq m} \phi(\chi) = \frac{\sum_{N(\mathbf{f})\leq m} \left(\sum_{\mathbf{f}_\chi = \mathbf{f}} \phi(\chi) \right) / \sum_{\mathbf{f}_\chi = \mathbf{f}} 1}{\sum_{N(\mathbf{f})\leq m} 1}$$

References

Y. Ihara, V. Kumar Murty, and Mahoro Shimura, *On the logarithmic derivatives of Dirichlet L-functions at* $s = 1$, Acta Arithmetica, vol 137, 2009.

Y. Ihara, *On the Euler-Kronecker constants of global fields and primes with small norms*, Algebraic Geometry and Number Theory, Progr. Math. (Birkhäuser Boston, Boston, MA), Vol. 253, pp. 407–451, 2006.

Keiju Sono, *Upper bounds for the moments of derivatives of Dirichlet L-functions*, Central European Journal of Mathematics, Vol 12(6), 2014.

Thank you!