A simple proof of the Wiener-Ikehara Tauberian theorem

Jagannath Sahoo (IIT Gandhinagar)
Joint work with M. Ram Murty (Queen's University) and Akshaa Vatwani (IIT Gandhinagar)

Comparative Prime Number Theory Symposium
University of British Columbia
Vancouver, BC, Canada
June 17–21, 2024

Abel's theorem

- Let
$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$
be a power series with $a_n \in \mathbb{R}$ which converges for $x \in (-1, 1)$. In 1826, Abel proved that if
$$\sum_{n=0}^{\infty} a_n = A,$$
then $\lim_{x \to 1^-} f(x)$ exists and equals to A.
- Is the converse true? No.
Eg. $\lim_{x \to 1^-} \sum_{n=0}^{\infty} (-1)^n x^n = \lim_{x \to 1^-} \frac{1}{1+x}$, but $\sum_{n=0}^{\infty} (-1)^n$ does not converge.

Abel's theorem

- Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

be a power series with $a_n \in \mathbb{R}$ which converges for $x \in (-1, 1)$. In 1826, Abel proved that if

$$\sum_{n=0}^{\infty} a_n = A,$$

then $\lim_{x \to 1^-} f(x)$ exists and equals to A.

- Is the converse true? No.

Eg. $\lim_{x \to 1^-} \sum_{n=0}^{\infty} (-1)^n x^n = \lim_{x \to 1^-} \frac{1}{1+x}$, but $\sum_{n=0}^{\infty} (-1)^n$ does not converge.

Abel's theorem

- Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

be a power series with $a_n \in \mathbb{R}$ which converges for $x \in (-1, 1)$. In 1826, Abel proved that if

$$\sum_{n=0}^{\infty} a_n = A,$$

then $\lim_{x \to 1^-} f(x)$ exists and equals to A.

- Is the converse true? No.

Eg. $\lim_{x \to 1^-} \sum_{n=0}^{\infty} (-1)^n x^n = \lim_{x \to 1^-} \frac{1}{1+x}$, but $\sum_{n=0}^{\infty} (-1)^n$ does not converge.

Abel's theorem

- Let
$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$
be a power series with $a_n \in \mathbb{R}$ which converges for $x \in (-1, 1)$. In 1826, Abel proved that if
$$\sum_{n=0}^{\infty} a_n = A,$$
then $\lim_{x \to 1^-} f(x)$ exists and equals to A.
- Is the converse true? No.

  Eg. $\lim_{x \to 1^-} \sum_{n=0}^{\infty} (-1)^n x^n = \lim_{x \to 1^-} \frac{1}{1+x}$, but $\sum_{n=0}^{\infty} (-1)^n$ does not converge.

- **A. Tauber (1897)**: Proved that if

$$\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = A$$

and $\lim_{n\to\infty} na_n = 0$, then

$$\sum_{n=0}^{\infty} a_n = A.$$

- **Wiener-Ikehara (1931)**: Obtained an asymptotic formula for the partial sum of coefficients of a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ satisfying certain conditions.

- **A. Tauber (1897)**: Proved that if

$$\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = A$$

and $\lim_{n \to \infty} n a_n = 0$, then

$$\sum_{n=0}^{\infty} a_n = A.$$

- **Wiener-Ikehara (1931)**: Obtained an asymptotic formula for the partial sum of coefficients of a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ satisfying certain conditions.

The Wiener-Ikehara Tauberian Theorem

Let

$$G(s) = \sum_{t=1}^{\infty} \frac{b_t}{t^s} \quad (b_t \geq 0)$$

be a Dirichlet series such that

- $G(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$,
- $G(s)$ can be analytically continued to $\mathrm{Re}(s) \geq 1$, except for a simple pole at $s = 1$ with residue R.

Then as $x \to \infty$,

$$\sum_{t \leq x} b_t \sim Rx.$$

The Wiener-Ikehara Tauberian Theorem

Let

$$G(s) = \sum_{t=1}^{\infty} \frac{b_t}{t^s} \quad (b_t \geq 0)$$

be a Dirichlet series such that

- $G(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$,
- $G(s)$ can be analytically continued to $\mathrm{Re}(s) \geq 1$, except for a simple pole at $s = 1$ with residue R.

Then as $x \to \infty$,

$$\sum_{t \leq x} b_t \sim Rx.$$

The Wiener-Ikehara Tauberian Theorem

Let

$$G(s) = \sum_{t=1}^{\infty} \frac{b_t}{t^s} \quad (b_t \geq 0)$$

be a Dirichlet series such that

- $G(s)$ is absolutely convergent for $\text{Re}(s) > 1$,
- $G(s)$ can be analytically continued to $\text{Re}(s) \geq 1$, except for a simple pole at $s = 1$ with residue R.

Then as $x \to \infty$,

$$\sum_{t \leq x} b_t \sim Rx.$$

Higher order poles

Let

$$G(s) = \sum_{t=1}^{\infty} \frac{b_t}{t^s} \quad (b_t \geq 0)$$

be a Dirichlet series such that

- $G(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$,
- $G(s)$ can be analytically continued to $\mathrm{Re}(s) \geq 1$, except for a pole of order k at $s = 1$.

Then, we have

$$\sum_{t \leq x} b_t = \frac{c_k}{(k-1)!} x(\log x)^{k-1} + O\left(x(\log x)^{k-2}\right),$$

as $x \to \infty$, where c_k is the residue of $(s-1)^{k-1} G(s)$ at $s = 1$.

Higher order poles

Let

$$G(s) = \sum_{t=1}^{\infty} \frac{b_t}{t^s} \quad (b_t \geq 0)$$

be a Dirichlet series such that

- $G(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$,
- $G(s)$ can be analytically continued to $\mathrm{Re}(s) \geq 1$, except for a pole of order k at $s = 1$.

Then, we have

$$\sum_{t \leq x} b_t = \frac{c_k}{(k-1)!} x(\log x)^{k-1} + O\left(x(\log x)^{k-2}\right),$$

as $x \to \infty$, where c_k is the residue of $(s-1)^{k-1}G(s)$ at $s = 1$.

Higher order poles

Let

$$G(s) = \sum_{t=1}^{\infty} \frac{b_t}{t^s} \quad (b_t \geq 0)$$

be a Dirichlet series such that

- $G(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$,
- $G(s)$ can be analytically continued to $\mathrm{Re}(s) \geq 1$, except for a pole of order k at $s = 1$.

Then, we have

$$\sum_{t \leq x} b_t = \frac{c_k}{(k-1)!} x (\log x)^{k-1} + O\left(x (\log x)^{k-2} \right),$$

as $x \to \infty$, where c_k is the residue of $(s-1)^{k-1} G(s)$ at $s = 1$.

In general

Let

$$F(s) = \sum_{t=1}^{\infty} \frac{a_t}{t^s}, \quad a_t \in \mathbb{C},$$

be a Dirichlet series such that

1. $F(s)$ is absolutely convergent for $\mathrm{Re}(s) > c, \quad (c > 0)$
2. $F(s)$ can be analytically continued to $\mathrm{Re}(s) \geq c$, except for a pole of order k at $s = c$,
3. $|a_t| \leq b_t$, where $b_t \geq 0$ and $G(s) = \sum_{t=1}^{\infty} \frac{b_t}{t^s}$ satisfies conditions (1) and (2) above.

Then, as $x \to \infty$, we have

$$\sum_{t \leq x} a_t = \frac{c_k}{c(k-1)!} x^c (\log x)^{k-1} + O\left(x^c (\log x)^{k-2}\right),$$

where c_k is the residue of $(s-c)^{k-1} F(s)$ at $s = c$.

In general

Let

$$F(s) = \sum_{t=1}^{\infty} \frac{a_t}{t^s}, \quad a_t \in \mathbb{C},$$

be a Dirichlet series such that

1. $F(s)$ is absolutely convergent for $\mathrm{Re}(s) > c, \quad (c > 0)$
2. $F(s)$ can be analytically continued to $\mathrm{Re}(s) \geq c$, except for a pole of order k at $s = c$,
3. $|a_t| \leq b_t$, where $b_t \geq 0$ and $G(s) = \sum_{t=1}^{\infty} \frac{b_t}{t^s}$ satisfies conditions (1) and (2) above.

Then, as $x \to \infty$, we have

$$\sum_{t \leq x} a_t = \frac{c_k}{c(k-1)!} x^c (\log x)^{k-1} + O\left(x^c (\log x)^{k-2}\right),$$

where c_k is the residue of $(s-c)^{k-1} F(s)$ at $s = c$.

Prime Number Theorem from the W-I theorem

- The Wiener-Ikehara theorem when applied to $-\zeta^{'}(s)/\zeta(s)$ yields the Prime Number Theorem.
- The logarithmic derivative of $\zeta(s)$, for $\mathrm{Re}(s) > 1$, is represented by the Dirichlet series

$$\frac{\zeta^{'}(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

- The above Dirichlet series satisfies the hypothesis of the Wiener-Ikehara Theorem with a simple pole of residue 1 at $s = 1$. Applying the Tauberian theorem yields

$$\sum_{n \leq x} \Lambda(n) \sim x.$$

Prime Number Theorem from the W-I theorem

- The Wiener-Ikehara theorem when applied to $-\zeta'(s)/\zeta(s)$ yields the Prime Number Theorem.
- The logarithmic derivative of $\zeta(s)$, for $\mathrm{Re}(s) > 1$, is represented by the Dirichlet series
$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$
- The above Dirichlet series satisfies the hypothesis of the Wiener-Ikehara Theorem with a simple pole of residue 1 at $s = 1$. Applying the Tauberian theorem yields
$$\sum_{n \leq x} \Lambda(n) \sim x.$$

Towards the proof

Let F be a smooth compactly supported function on $\mathbb{R}$ and η_F be the Fourier transform of $F(t)e^t$:

$$F(t)e^t = \int_{\mathbb{R}} \eta_F(u)e^{-iut}\, du.$$

- In particular,

$$F\left(\frac{\log t}{\log x}\right) = \int_{\mathbb{R}} \eta_F(u)t^{-\frac{1+iu}{\log x}}\, du.$$

- We have

$$|\eta_F(t)| \ll_A \frac{1}{(1+|t|)^A},$$

for any $A > 0$, as $|t| \to \infty$.

Towards the proof

Let F be a smooth compactly supported function on $\mathbb{R}$ and η_F be the Fourier transform of $F(t)e^t$:

$$F(t)e^t = \int_{\mathbb{R}} \eta_F(u)e^{-iut}\,du.$$

- In particular,

$$F\left(\frac{\log t}{\log x}\right) = \int_{\mathbb{R}} \eta_F(u)t^{-\frac{1+iu}{\log x}}\,du.$$

- We have

$$|\eta_F(t)| \ll_A \frac{1}{(1+|t|)^A},$$

  for any $A > 0$, as $|t| \to \infty$.

Preliminary lemma

Let

$$\int^{(j)} F(t)\,dt := \int_{t_j=0}^{\infty} \int_{t_{j-1}\geq t_j} \cdots \int_{t_1 \geq t_2} F(t_1)dt_1 \ldots dt_j.$$

We will require the identity

$$\begin{aligned} \int_{\mathbb{R}} \eta_F(u) \frac{1}{(1+iu)^j}\,du &= \int^{(j)} F(t)\,dt \\ &= \frac{1}{(j-1)!} \int_0^{\infty} F(t) t^{j-1}\,dt. \end{aligned}$$

Proof of the W-I Theorem with higher order pole

- In order to estimate $\sum_{t \leq x} b_t$, we will consider the related sum $\sum_{t \leq x} \frac{b_t}{t}$ and then use partial summation.
- In order to estimate the sum $\sum_{t \leq x} \frac{b_t}{t}$, we consider the infinite series $\sum_{t=1}^{\infty} \frac{b_t}{t}$, twisted by an appropriate smooth bump function $F(t)$ which approximates the indicator function $\mathbf{1}_{[1,x]}$.
- More precisely, for some $\delta > 0$ to be chosen later, we take a smooth function $F(t)$ supported on $[-\delta, 1+\delta]$ such that $F(t) = 1$ for $0 \leq t \leq 1$.
- Essentially

$$\sum_{t \leq x} \frac{b_t}{t} \approx \sum_{t=1}^{\infty} \frac{b_t}{t} F\left(\frac{\log t}{\log x}\right).$$

Proof of the W-I Theorem with higher order pole

- In order to estimate $\sum_{t \leq x} b_t$, we will consider the related sum $\sum_{t \leq x} \frac{b_t}{t}$ and then use partial summation.
- In order to estimate the sum $\sum_{t \leq x} \frac{b_t}{t}$, we consider the infinite series $\sum_{t=1}^{\infty} \frac{b_t}{t}$, twisted by an appropriate smooth bump function $F(t)$ which approximates the indicator function $\mathbf{1}_{[1,x]}$.
- More precisely, for some $\delta > 0$ to be chosen later, we take a smooth function $F(t)$ supported on $[-\delta, 1+\delta]$ such that $F(t) = 1$ for $0 \leq t \leq 1$.
- Essentially

$$\sum_{t \leq x} \frac{b_t}{t} \approx \sum_{t=1}^{\infty} \frac{b_t}{t} F\left(\frac{\log t}{\log x}\right).$$

Proof of the W-I Theorem with higher order pole

- In order to estimate $\sum_{t \leq x} b_t$, we will consider the related sum $\sum_{t \leq x} \frac{b_t}{t}$ and then use partial summation.
- In order to estimate the sum $\sum_{t \leq x} \frac{b_t}{t}$, we consider the infinite series $\sum_{t=1}^{\infty} \frac{b_t}{t}$, twisted by an appropriate smooth bump function $F(t)$ which approximates the indicator function $\mathbf{1}_{[1,x]}$.
- More precisely, for some $\delta > 0$ to be chosen later, we take a smooth function $F(t)$ supported on $[-\delta, 1+\delta]$ such that $F(t) = 1$ for $0 \leq t \leq 1$.
- Essentially

$$\sum_{t \leq x} \frac{b_t}{t} \approx \sum_{t=1}^{\infty} \frac{b_t}{t} F\left(\frac{\log t}{\log x}\right).$$

Proof of the W-I Theorem with higher order pole

- In order to estimate $\sum_{t\leq x} b_t$, we will consider the related sum $\sum_{t\leq x} \frac{b_t}{t}$ and then use partial summation.
- In order to estimate the sum $\sum_{t\leq x} \frac{b_t}{t}$, we consider the infinite series $\sum_{t=1}^{\infty} \frac{b_t}{t}$, twisted by an appropriate smooth bump function $F(t)$ which approximates the indicator function $\mathbf{1}_{[1,x]}$.
- More precisely, for some $\delta > 0$ to be chosen later, we take a smooth function $F(t)$ supported on $[-\delta, 1+\delta]$ such that $F(t) = 1$ for $0 \leq t \leq 1$.
- Essentially

$$\sum_{t\leq x} \frac{b_t}{t} \approx \sum_{t=1}^{\infty} \frac{b_t}{t} F\left(\frac{\log t}{\log x}\right).$$

- Recalling $F\left(\frac{\log t}{\log x}\right) = \int_{\mathbb{R}} \eta_F(u) t^{-\frac{1+iu}{\log x}}\, du$, we have

$$\sum_{t=1}^{\infty} \frac{b_t}{t} F\left(\frac{\log t}{\log x}\right) = \int_{\mathbb{R}} \eta_F(u) \sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1+iu}{\log x}}} du,$$

  interchanging the order of summation and integration in the region of absolute convergence of the Dirichlet series.

- For $s \to 0^+$, we will use the Laurent series expansion:

$$G(1+s) = \frac{c_k}{s^k} + \frac{c_{k-1}}{s^{k-1}} + O\left(\frac{1}{|s|^{k-2}}\right),$$

  where c_i is the residue of $(s-1)^{i-1} G(s)$ at $s = 1$.

- Recalling $F\left(\frac{\log t}{\log x}\right) = \int_{\mathbb{R}} \eta_F(u) t^{-\frac{1+iu}{\log x}}\, du$, we have

$$\sum_{t=1}^{\infty} \frac{b_t}{t} F\left(\frac{\log t}{\log x}\right) = \int_{\mathbb{R}} \eta_F(u) \sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1+iu}{\log x}}} du,$$

  interchanging the order of summation and integration in the region of absolute convergence of the Dirichlet series.

- For $s \to 0^+$, we will use the Laurent series expansion:

$$G(1+s) = \frac{c_k}{s^k} + \frac{c_{k-1}}{s^{k-1}} + O\left(\frac{1}{|s|^{k-2}}\right),$$

  where c_i is the residue of $(s-1)^{i-1} G(s)$ at $s = 1$.

Fix $\epsilon \in (0,1)$. Case 1: $|u| < (\log x)^{\epsilon}$

The contribution to the integral

$$\int_{\mathbb{R}} \eta_F(u) \sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1+iu}{\log x}}} du$$

from this region is

$$\sum_{j=k-1}^{k} c_j (\log x)^j \int\limits_{|u|<(\log x)^{\epsilon}} \frac{\eta_F(u)}{(1+iu)^j} du + O\left((\log x)^{k-2} \int_{\mathbb{R}} \frac{|\eta_F(u)|}{|1+iu|^{k-2}} du \right).$$

The error term is $\ll (\log x)^{k-2}$ due to the rapid decay of $\eta_F(u)$.

Case 2: $|u| \geq (\log x)^{\epsilon}$

Here one cannot directly use the Laurent series expansion. But we have

$$\left| \sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1+iu}{\log x}}} \right| \leq \sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1}{\log x}}}$$

$$\ll (\log x)^k,$$

using the Laurent series expansion for $G(1+(\log x)^{-1})$ as $x \to \infty$.
The contribution to

$$\int_{\mathbb{R}} \eta_F(u) \sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1+iu}{\log x}}} du$$

is thus

$$\ll (\log x)^k \int_{|u| \geq (\log x)^{\epsilon}} \eta_F(u) du \ll_A (\log x)^{-A},$$

for any $A > 0$ since in this region we have

$$|\eta_F(u)| \ll_A \frac{1}{(1+|u|)^{3A}} \ll_A (\log x)^{-k-A} \frac{1}{(1+|u|)^A}.$$

Case 2: $|u| \geq (\log x)^\epsilon$

Here one cannot directly use the Laurent series expansion. But we have

$$\left|\sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1+iu}{\log x}}}\right| \leq \sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1}{\log x}}} \ll (\log x)^k,$$

using the Laurent series expansion for $G(1+(\log x)^{-1})$ as $x \to \infty$.

The contribution to

$$\int_{\mathbb{R}} \eta_F(u) \sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1+iu}{\log x}}} du$$

is thus

$$\ll (\log x)^k \int_{|u| \geq (\log x)^\epsilon} \eta_F(u) du \ll_A (\log x)^{-A},$$

for any $A > 0$ since in this region we have

$$|\eta_F(u)| \ll_A \frac{1}{(1+|u|)^{3A}} \ll_A (\log x)^{-k-A} \frac{1}{(1+|u|)^A}.$$

Case 2: $|u| \geq (\log x)^{\epsilon}$

Here one cannot directly use the Laurent series expansion. But we have

$$\left|\sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1+iu}{\log x}}}\right| \leq \sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1}{\log x}}} \ll (\log x)^k,$$

using the Laurent series expansion for $G(1+(\log x)^{-1})$ as $x \to \infty$.
The contribution to

$$\int_{\mathbb{R}} \eta_F(u) \sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1+iu}{\log x}}} du$$

is thus

$$\ll (\log x)^k \int_{|u| \geq (\log x)^{\epsilon}} \eta_F(u) du \ll_A (\log x)^{-A},$$

for any $A > 0$ since in this region we have

$$|\eta_F(u)| \ll_A \frac{1}{(1+|u|)^{3A}} \ll_A (\log x)^{-k-A} \frac{1}{(1+|u|)^A}.$$

Case 2: $|u| \geq (\log x)^{\epsilon}$

Here one cannot directly use the Laurent series expansion. But we have

$$\left|\sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1+iu}{\log x}}}\right| \leq \sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1}{\log x}}}$$
$$\ll (\log x)^k,$$

using the Laurent series expansion for $G(1+(\log x)^{-1})$ as $x \to \infty$.
The contribution to

$$\int_{\mathbb{R}} \eta_F(u) \sum_{t=1}^{\infty} \frac{b_t}{t^{1+\frac{1+iu}{\log x}}} du$$

is thus

$$\ll (\log x)^k \int_{|u| \geq (\log x)^{\epsilon}} \eta_F(u) du \ll_A (\log x)^{-A},$$

for any $A > 0$ since in this region we have

$$|\eta_F(u)| \ll_A \frac{1}{(1+|u|)^{3A}} \ll_A (\log x)^{-k-A} \frac{1}{(1+|u|)^{A}}.$$

Combining Cases

We have obtained

$$\sum_{t=1}^{\infty} \frac{b_t}{t} F\left(\frac{\log t}{\log x}\right) = \sum_{j=k-1}^{k} c_j (\log x)^j \int_{|u|<(\log x)^{\epsilon}} \frac{\eta_F(u)}{(1+iu)^j} du + O\left((\log x)^{k-2}\right).$$

One may replace the integral above by the full integral since

$$\int_{|u|\geq(\log x)^{\epsilon}} \frac{\eta_F(u)}{(1+iu)^j} du \ll_A (\log x)^{-A},$$

by the rapid decay of η_F.

Combining Cases

We have obtained

$$\sum_{t=1}^{\infty} \frac{b_t}{t} F\left(\frac{\log t}{\log x}\right) = \sum_{j=k-1}^{k} c_j (\log x)^j \int_{|u|<(\log x)^{\epsilon}} \frac{\eta_F(u)}{(1+iu)^j} du + O\left((\log x)^{k-2}\right).$$

One may replace the integral above by the full integral since

$$\int_{|u|\geq(\log x)^{\epsilon}} \frac{\eta_F(u)}{(1+iu)^j} du \ll_A (\log x)^{-A},$$

by the rapid decay of η_F.

- Thus

$$\sum_{t=1}^{\infty} \frac{b_t}{t} F\left(\frac{\log t}{\log x}\right) = \sum_{j=k-1}^{k} c_j (\log x)^j \int_{\mathbb{R}} \frac{\eta_F(u)}{(1+iu)^j} du + O\left((\log x)^{k-2}\right).$$

- Use the Preliminary lemma to move from $\eta_F(u)$ back to the function $F(t)$. This yields

$$\sum_{t=1}^{\infty} \frac{b_t}{t} F\left(\frac{\log t}{\log x}\right) = \sum_{j=k-1}^{k} c_j \frac{(\log x)^j}{(j-1)!} \int_0^{\infty} F(t) t^{j-1} dt + O\left((\log x)^{k-2}\right).$$

- Thus

$$\sum_{t=1}^{\infty} \frac{b_t}{t} F\left(\frac{\log t}{\log x}\right) = \sum_{j=k-1}^{k} c_j (\log x)^j \int_{\mathbb{R}} \frac{\eta_F(u)}{(1+iu)^j} du + O\left((\log x)^{k-2}\right).$$

- Use the Preliminary lemma to move from $\eta_F(u)$ back to the function $F(t)$. This yields

$$\sum_{t=1}^{\infty} \frac{b_t}{t} F\left(\frac{\log t}{\log x}\right) = \sum_{j=k-1}^{k} c_j \frac{(\log x)^j}{(j-1)!} \int_0^{\infty} F(t) t^{j-1} dt + O\left((\log x)^{k-2}\right).$$

- Recall that $F(t)$ was a smooth bump function supported on $[-\delta, 1+\delta]$. Choose

$$0 < \delta < \log\left(\frac{\lfloor x \rfloor + 1}{x}\right)(\log x)^{-k}.$$

With this choice of δ, we obtain

$$\sum_{t \le x} \frac{b_t}{t} = \frac{c_k (\log x)^k}{k!} + \frac{c_{k-1}(\log x)^{k-1}}{(k-1)!} + O\left((\log x)^{k-2}\right).$$

- Applying partial summation and integration by parts yields

$$\sum_{t \le x} b_t = \frac{c_k}{(k-1)!} x (\log x)^{k-1} + O\left(x(\log x)^{k-2}\right),$$

as needed. □

- Recall that $F(t)$ was a smooth bump function supported on $[-\delta, 1+\delta]$. Choose

$$0 < \delta < \log\left(\frac{\lfloor x \rfloor + 1}{x}\right)(\log x)^{-k}.$$

With this choice of δ, we obtain

$$\sum_{t \le x} \frac{b_t}{t} = \frac{c_k(\log x)^k}{k!} + \frac{c_{k-1}(\log x)^{k-1}}{(k-1)!} + O\left((\log x)^{k-2}\right).$$

- Applying partial summation and integration by parts yields

$$\sum_{t \le x} b_t = \frac{c_k}{(k-1)!} x(\log x)^{k-1} + O\left(x(\log x)^{k-2}\right),$$

as needed. □

- Recall that $F(t)$ was a smooth bump function supported on $[-\delta, 1+\delta]$. Choose

$$0 < \delta < \log\left(\frac{\lfloor x \rfloor + 1}{x}\right)(\log x)^{-k}.$$

With this choice of δ, we obtain

$$\sum_{t \leq x} \frac{b_t}{t} = \frac{c_k (\log x)^k}{k!} + \frac{c_{k-1}(\log x)^{k-1}}{(k-1)!} + O\left((\log x)^{k-2}\right).$$

- Applying partial summation and integration by parts yields

$$\sum_{t \leq x} b_t = \frac{c_k}{(k-1)!} x (\log x)^{k-1} + O\left(x (\log x)^{k-2}\right),$$

as needed. □

Concluding remarks

If $G(s)$ has the Laurent series expansion

$$G(1+s) = \frac{c_k}{s^k} + \frac{c_{k-1}}{s^{k-1}} + \cdots + c_0 + O_m\big(|s|^m\big),$$

as $s \to 0^+$, for some $c_i \in \mathbb{C}$ and $m \in \mathbb{N}$, then we obtain a saving of $(\log x)^m$ in the error term. More precisely, we obtain

$$\sum_{t \le x} \frac{b(t)}{t} = c_k \frac{(\log x)^k}{k!} + c_{k-1} \frac{(\log x)^{k-1}}{(k-1)!} + \cdots + c_0 + O_m\left(\frac{1}{(\log x)^m}\right).$$

By applying partial summation, one can derive

$$\sum_{t \le x} b(t) = x \sum_{j=1}^{k} \frac{(\log x)^{k-j}}{(k-j)!} \lambda_{k-j} + O_m\left(\frac{x}{(\log x)^m}\right),$$

where $\lambda_{k-j} = \sum_{i=0}^{j-1} (-1)^{j-1-i} c_{k-i}$.

Concluding remarks

If $G(s)$ has the Laurent series expansion

$$G(1+s) = \frac{c_k}{s^k} + \frac{c_{k-1}}{s^{k-1}} + \cdots + c_0 + O_m\big(|s|^m\big),$$

as $s \to 0^+$, for some $c_i \in \mathbb{C}$ and $m \in \mathbb{N}$, then we obtain a saving of $(\log x)^m$ in the error term. More precisely, we obtain

$$\sum_{t \leq x} \frac{b(t)}{t} = c_k \frac{(\log x)^k}{k!} + c_{k-1} \frac{(\log x)^{k-1}}{(k-1)!} + \cdots + c_0 + O_m\left(\frac{1}{(\log x)^m}\right).$$

By applying partial summation, one can derive

$$\sum_{t \leq x} b(t) = x \sum_{j=1}^{k} \frac{(\log x)^{k-j}}{(k-j)!} \lambda_{k-j} + O_m\left(\frac{x}{(\log x)^m}\right),$$

where $\lambda_{k-j} = \sum_{i=0}^{j-1} (-1)^{j-1-i} c_{k-i}$.

References

- S Ikehara, An extension of Landau's theorem in the analytic theory of numbers,Journal of Math. and Phys. of the Mass. Inst. of Technology, 10 (1931), 1-12.
- J. Korevaar, A century of complex Tauberian theory, Bull. Amer. Math. Soc., 39(4) (2002), 475-531.
- M. Ram Murty, Problems in Analytic number theory, 2nd Edition, Graduate Texts in Mathematics, Springer, New York, 2008.
- A. Tauber, Ein Satz aus der Theorie der unendlichen Reihen, Monatshefte f. Math.,8 (1897), 273-277.
- G. Tenenbaum, Introduction to analytic and probabilistic number theory, third ed., Graduate Studies in Mathematics, vol. 163, American Mathematical Society, Providence, RI, 2015.

THANK YOU