

# Quadratic twists of modular $L$ -functions

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## Quadratic twists

Let  $f$  be a primitive modular form of weight  $\kappa$  and level  $N$  and suppose that  $f$  is a Hecke eigenform. The  $L$ -function associated with  $f$  is given by

$$L(s, f) = \sum_n \frac{\lambda_f(n)}{n^s},$$

for  $\operatorname{Re} s > 1$ , and can be analytically continued to the entire complex plane.

## Quadratic twists

Let  $d$  be a fundamental discriminant, and  $\chi_d(\cdot) = \left(\frac{d}{\cdot}\right)$  denote the primitive quadratic character with conductor  $|d|$ . Then  $f \otimes \chi_d$  is a Hecke eigenform, with  $L$ -function given by

$$L(s, f \otimes \chi_d) = \sum_n \frac{\lambda_f(n)\chi_d(n)}{n^s} \quad (1)$$

for  $\operatorname{Re} s > 1$ .

$$x^2 + y^2 + z^2 = n$$

$n$  is represented by  $x^2 + y^2 + z^2$  if it is not of the form  $4^a(8b - 1)$ .  
When representations exist, one may ask whether the points  $\frac{1}{\sqrt{n}}(x, y, z)$  equidistribute on the unit sphere.

- Linnik resolved this using ergodic methods subject to some condition that  $n$  is a quadratic residue modulo some prime.
- Iwaniec resolved this without the assumption by bounding Fourier coefficients of half weight integer modular forms.
- By Waldspurger's formula, one may relate the Fourier coefficient of a primitive half integer weight cusp form  $g$  to  $L$ -values of a quadratic twist of a modular form  $f$ .

$$|\rho_g(|D|)|^2 \asymp L(1/2, f \otimes \chi_D),$$

for  $D$  fundamental discriminant.

## Quadratic twists of elliptic curves

We call a natural number  $n$  congruent if  $n$  occurs as the area of a right angle triangle with sides of rational length. This gives a system

$$a^2 + b^2 = c^2, \text{ and } \frac{ab}{2} = n.$$

A change of variables  $x = n(a + c)/b$  and  $y = 2n^2(a + c)/b^2$  gives

$$y^2 = x^3 - n^2x,$$

and  $n$  is congruent if and only if there are solutions for the above in rational  $x, y$  with  $y \neq 0$ . A change of variables  $(x, y) \rightarrow (nx, n^2y)$  gives the equivalent form

$$ny^2 = x^3 - x.$$

These are the quadratic twists of

$$y^2 = x^3 - x.$$

More generally when an elliptic curve  $E$  is given by  $y^2 = x^3 + ax + b$ , the quadratic twist  $E_d$  is given by

$$dy^2 = x^3 + ax + b.$$

For  $L(s, E)$  the  $L$ -function of  $E$ ,  $L(s, E_d) = L(s, E \otimes \chi_d)$  (in the sense previously introduced).

The rational points on  $E_d$  is an abelian group of finite rank  $r$ . The Birch and Swinnerton-Dyer predicts that  $r$  is the same as the order of vanishing of  $L(s, E_d)$  at the critical point.

## Moments

For simplicity, assume  $f$  is full level and restrict attention to fundamental discriminants of the form  $8d$  where  $d$  is odd and squarefree. We let  $\sum^*$  denote a sum over squarefree integers. It is of high interest to understand moments of the form

$$M(k) := \sum_{\substack{0 < 8d < X \\ (d, 2) = 1}}^* L(1/2, f \otimes \chi_{8d})^k. \quad (2)$$

## Moments

Keating and Snaith conjectured that

$$M(k) \sim C(k, f) X(\log X)^{\frac{k(k-1)}{2}},$$

for an explicit constant  $C(k, f)$ . This conjecture is analogous to conjectures for moments of other families.

- The conjecture is known for the first moment  $k = 1$  by Iwaniec's work.
- Based on knowledge of the twisted first moment, Radziwiłł and Soundararajan proved that  $M(k) \ll X(\log X)^{\frac{k(k-1)}{2}}$  for  $0 \leq k \leq 1$ .



## The second moment

For  $k = 2$ ,

- The work of Heath-Brown implies that  $M(2) \ll X^{1+\epsilon}$ .
- The method of Soundararajan gives  $M(2) \ll X(\log X)^{1+\epsilon}$ , and refinement by Harper gives  $M(2) \ll X(\log X)$ , both conditionally on GRH.
- Based on similar ideas applied to bounding shifted moments, Soundararajan and Young proved the conjectured asymptotic for  $M(2)$  assuming GRH.

## The second moment

### Theorem

$$\sum_{\substack{0 < 8d < X \\ (d,2)=1}}^* L(1/2, f \otimes \chi_{8d})^2 \sim C_f X \log X,$$

where  $C_f$  is some explicit constant depending on  $f$ .

If we include a smooth weight in the sum over  $d$  above, the result can be proven with an error term of quality  $O(X(\log X)^{1/2+\epsilon})$  and improved to  $O(X(\log X)^\epsilon)$  with a little effort.

# Moments of quadratic Dirichlet $L$ -functions

Let

$$\mathcal{M}(k) = \sum_{\substack{0 < 8d < X \\ (d, 2) = 1}}^* L(1/2, \chi_{8d})^k.$$

- The first and second moments were computed by Jutila, and the third moment by Soundararajan.
- Refinements with improved error terms: on the first and third moments by Young, the second moment by Sono, and a further refinement of the third moment by Diaconu and Whitehead explicating a power saving secondary term.
- The fourth moment was computed recently assuming GRH by Shen, following the approach of Soundararajan and Young.

# Moments of quadratic Dirichlet $L$ -functions

Let

$$\mathcal{M}(k) = \sum_{\substack{0 < 8d < X \\ (d, 2) = 1}}^* L(1/2, \chi_{8d})^k.$$

- Florea gave the expected asymptotic for the analogous fourth moment over the function field  $\mathbb{F}_q[x]$  (where the Riemann hypothesis is known) with the base field  $\mathbb{F}_q$  fixed and genus going to infinity.
- Our techniques should extend to give the asymptotic for  $\mathcal{M}(4)$  unconditionally. This is work in progress by Shen and Stucky.

## Moments of quadratic twists and rank

Let  $m_d$  be the order of vanishing of  $L(s, f \otimes \chi_{8d})$  at  $s = 1/2$ , and let

$$R(X) = \sum_{\substack{0 < 8d < X \\ (d,2)=1}}^* m_d.$$

- Goldfeld proved that  $R(X) \ll X$  conditionally on GRH.
- Trivially,  $R(X) \ll X \log X$ , while the work of Perelli and Pomykala gives  $R(X) = o(X \log X)$ .
- Our methods may yield  $R(X) \ll X \log \log X$  proceeding along the same lines.
- Related: Mallesham Kummari is adapting these methods to derive asymptotics for moments of derivatives of the quadratic twist of modular  $L$  functions.

## Reduction to Dirichlet polynomials

After an application of the approximate functional equation, we morally need to understand sums like

$$\sum_{m \asymp X}^* \left| \sum_{n \ll X} \frac{\lambda_f(n)}{\sqrt{n}} \left( \frac{m}{n} \right) \right|^2. \quad (3)$$

## Poisson and functional equation

We have two basic tools. Roughly,

- Poisson summation changes a sum like  $\sum_{m \asymp M} \left( \frac{m}{n_1 n_2} \right)$  into a dual sum of length  $n_1 n_2 / M$ .
- Functional equation changes  $\sum_{n \asymp N} \frac{\lambda_f(n)}{\sqrt{n}} \left( \frac{m}{n} \right)$  into a similar dual sum of length  $|m|^2 / N$ .

General rule of thumb is that we prefer shorter sums.  
Neither tool seems to help us understand

$$\sum_{m \asymp X}^* \left| \sum_{n \asymp X} \frac{\lambda_f(n)}{\sqrt{n}} \left( \frac{m}{n} \right) \right|^2.$$

## Truncation

The functional equation and Poisson summation are useful in the easier range

$$\sum_{m \asymp X}^* \left| \sum_{n \ll X/(\log^A X)} \frac{\lambda_f(n)}{\sqrt{n}} \left(\frac{m}{n}\right) \right|^2, \quad (4)$$

for some large  $A > 0$ . Thus, the challenge is to bound sums of the form

$$S = \sum_{m \asymp X}^* \left| \sum_{n \asymp N} \frac{\lambda_f(n)}{\sqrt{n}} \left(\frac{m}{n}\right) \right|^2, \quad (5)$$

when  $N$  is close to  $X$ .



## Large sieve type bound I

We will show that

$$\sum_{m \asymp X}^* \left| \sum_{n \asymp N} \frac{\lambda_f(n)}{\sqrt{n}} \left( \frac{m}{n} \right) \right|^2 \ll X, \quad (6)$$

which is best possible up to the implied constant. Assuming this, dyadic summation for  $\frac{X}{\log^A X} \leq N \ll X$  gives the bound

$$\sum_{m \asymp X}^* \left| \sum_{X/(\log^A X) \ll n \ll X} \frac{\lambda_f(n)}{\sqrt{n}} \left( \frac{m}{n} \right) \right|^2 \ll X(\log X)^\epsilon.$$

This reduces the problem to considering (4).

## Large sieve type bound II

The main challenge is to prove

$$\sum_{m \asymp X}^* \left| \sum_{n \asymp X} a(n) \left( \frac{m}{n} \right) \right|^2 \ll X, \quad (7)$$

which is best possible up to the implied constant.  
Heath-Brown's work implies that

$$\sum_{m \asymp X}^* \left| \sum_{n \asymp X} a(n) \left( \frac{m}{n} \right) \right|^2 \ll X \cdot X^\epsilon, \quad (8)$$

and the work of Soundararajan and Young implicitly prove that

$$S \ll X(\log X)^{1/2+\epsilon}$$

conditionally on GRH.

# A sketch of a sketch

## Inflation I

For a prime  $p \asymp \sqrt{L}$

$$S = \sum_{m \asymp X}^* \left| \sum_{n \asymp X} a(n) \left( \frac{m}{n} \right) \right|^2 = \sum_{m \asymp X}^* \left| \sum_{\substack{n \asymp X \\ p \nmid n}} a(n) \left( \frac{mp^2}{n} \right) + \sum_{\substack{n \asymp X \\ p \mid n}} a(n) \left( \frac{m}{n} \right) \right|^2.$$

## Inflation II

Let  $\mathcal{P}(L) \asymp \frac{\sqrt{L}}{\log L}$  be the number of primes in the interval  $[\sqrt{L}, 2\sqrt{L}]$  and sum over all  $p \in [\sqrt{L}, 2\sqrt{L}]$  to see that

$$\begin{aligned} \mathcal{P}(L)S &\ll \sum_{\sqrt{L} \leq p \leq 2\sqrt{L}} \sum_{m \asymp X}^* \left| \sum_{n \asymp X} a(n) \left( \frac{mp^2}{n} \right) \right|^2 + \text{other} \\ &\leq \sum_{m \asymp 4XL} \left| \sum_{n \asymp X} a(n) \left( \frac{m}{n} \right) \right|^2 + \text{other}. \end{aligned}$$

## Remarks

- We have embedded our original sum over  $m$  into a longer sum, so that it is now advantageous to execute Poisson over  $m$  and begin the iterative process.
- Note that discarding the squarefree condition on  $m$  can be disastrous for arbitrary coefficients  $a(n)$ . We therefore expect to crucially use the special properties of  $a(n) = \frac{\lambda_f(n)}{\sqrt{n}}$ .
- We have used that  $\left(\frac{p^2}{n}\right)$  tends to be trivial, and that the representation of  $m$  by  $m'p^2$  is unique for  $m'$  squarefree.

## Poisson

Opening up the square and applying Poisson summation roughly gives that

$$\begin{aligned} & \sum_{m \asymp XL} \left| \sum_{n \asymp X} a(n) \left( \frac{m}{n} \right) \right|^2 \\ &= C_f XL + \frac{XL}{2} \sum_{n_1, n_2 \asymp X} \frac{\lambda_f(n_1) \lambda_f(n_2)}{\sqrt{n_1 n_2} n_1 n_2} \sum_{\substack{k \neq 0 \\ k \ll X^2 / XL \asymp X/L}} G_k(n_1 n_2). \end{aligned}$$

Generically,  $G_k(n_1 n_2)$  is  $\chi_k(n_1 n_2) \sqrt{n_1 n_2}$  when  $n_1 n_2$  is squarefree.

## Functional equation

The sum over  $k$  is essentially restricted to  $k \ll X^2/XL \asymp X/L$ , so we need to bound

$$\frac{XL}{2X} \sum_{k \asymp X/L} \sum_{n_1, n_2 \asymp X} \frac{\lambda_f(n_1)\lambda_f(n_2)}{\sqrt{n_1 n_2}} \frac{G_k(n_1 n_2)}{\sqrt{n_1 n_2}}. \quad (9)$$

Now we replace  $G_k(n_1 n_2)$  by  $\chi_k(n_1 n_2)\sqrt{n_1 n_2}$  so we hope to instead study a quantity like

$$L \sum_{k \asymp X/L}^* \left| \sum_{n \asymp X} \frac{\lambda_f(n)\chi_k(n)}{\sqrt{n}} \right|^2. \quad (10)$$

Since the conductor  $k \asymp X/L$  has been reduced, it now makes sense to apply the functional equation of  $L(s, f \otimes \chi_k)$  to transform the sum over  $n$  to a sum of length  $X/L^2$ .



## Structural comment

We expect

$$L \sum_{k \asymp X/L}^* \left| \sum_{n \asymp X/L^2} \frac{\lambda_f(n) \chi_k(n)}{\sqrt{n}} \right|^2 = CX \sum_{\substack{n_1, n_2 \asymp X/L^2 \\ n_1 n_2 = \square}} \frac{\lambda_f(n_1) \lambda_f(n_2)}{\sqrt{n_1 n_2}} + \text{small},$$

for some constant  $C$ .

- The "diagonal" contribution when  $n_1 n_2$  is a perfect square dominates.
- However, generically  $G_k(n_1 n_2) = 0$  when  $n_1 n_2$  is not squarefree, so that the same "diagonal" contribution does not exist in the prior sum.
- Careful analysis of the factors at prime squares and higher powers is crucial.

## “Many” prime squares

Our calculations suggest that

$$\mathcal{P}(L)S \leq CLX$$

for some constant  $C$ . Since  $\mathcal{P}(L) \asymp \sqrt{L}/\log L$ ,

$$S \leq L^{2/3}X.$$

Using this as our induction hypothesis,

$$\mathcal{P}(L)S \leq C_1XL + C_2L(L^{2/3}X/L) = (C_1 + C_2L^{-1/3})XL,$$

and so

$$S \leq L^{2/3}X \frac{C_1 + C_2L^{-1/3}}{L^{1/7}}.$$

So the number of primes in the interval  $[\sqrt{L}, 2\sqrt{L}]$  being large serves to control the loss of constant factors.

## Open problems

- $\sum_d^* L(1/2, f \otimes \chi_d)L(1/2, g \otimes \chi_d) \sim$
- $\sum_d^* L(1/2, f \otimes \chi_d)^2(\text{Short Poly})^2 \sim$

We have pretended that Poisson summation is useless unless the dual sum is shorter. But we have seen that the structure of the dual sum is different.