Moments of real Dirichlet L-functions and multiple Dirichlet series

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CRG weekly seminar on L-functions

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Moments of real Dirichlet L-functions

We will study the moments

$$\sum_{d \le X}^{*} L(1/2, \chi_d)^k,$$

where the sum runs over positive fundamental discriminants and $\chi_d = \left(\frac{d}{\cdot}\right)$.

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Shen (2021): Leading order term for $k = 4$ under GRH (based on work of Soundararajan and Young).

Shen, Stucky (2024): Several leading order terms for $k = 4$ unconditionally (based on work of X. Li).

Conjectures

Keating, Snaith (2000): conjectured the asymptotic formula

$$\sum_{d \leq X}^{*} L(1/2, \chi_d)^k \sim c_k X (\log X)^{k(k+1)/2}$$

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Conrey, Farmer, Keating, Rubinstein, Snaith (2005): came up with the recipe, which predicted that

$$\sum_{d \le X}^{*} L(1/2, \chi_d)^k = X P_k(\log X) + O(X^{1-\delta}),$$

where $P_k(x)$ is a (computable) polynomial of degree $k(k+1)/2$.

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where $P_k(x)$ is a (computable) polynomial of degree $k(k+1)/2$. They considered the shifted moments

$$\sum_{d\le X}^{*} L(s_1,\chi_d)\dots L(s_k,\chi_d),$$

where $s_j = 1/2+\alpha_j \approx \frac{1}{2}$.

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- Showed that if it has a meromorphic continuation beyond a particular point, then the conjecture of Keating and Snaith is true.
- Computed the third moment with an improved error term.
- Predicted a secondary term of size $X^{3/4}$ for the third moment (proved by Diaconu, Whitehead (2021)).

Moments of Dirichlet L-functions

We have

$$L(s,\chi)^k = \sum_{n\geq 1} \frac{\tau_k(n)\chi(n)}{n^s},$$

where $\tau_k(n) = (\underbrace{1 * \cdots * 1}_{k})(n)$.

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Then

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$$\sum_{d\leq X}^{*} \left(\frac{d}{n}\right) = \begin{cases} O(n^{1/2+\varepsilon}), & \text{if } n \neq \square, \\ \frac{X}{2\zeta(2)} a(n) + O(n^{1/2+\varepsilon}), & \text{if } n = \square, \end{cases}$$

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The approximate functional equation:

$$L(s,\chi_d) \approx \sum_{n\le x} \frac{\chi_d(n)}{n^s} + X_d(s) \sum_{n\le y} \frac{\chi_d(n)}{n^{1-s}},$$

where

$$L(s,\chi_d) = X_d(s) L(1-s,\chi_d), \qquad X_d(s) = d^{1/2-s} X(s), \qquad X(s) = \pi^{s-1/2} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

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Here $xy \asymp d$, it is important to have both parts.

The recipe for real Dirichlet L-functions

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A typical term after expanding the product is

$$\sum_{d\le X}^{*}\prod_{j\in J} X_d(s_j)\sum_{n_1,\dots,n_k\ge 1}\frac{\chi_d(n_1\dots n_k)}{n_1^{s_1^J}\dots n_k^{s_k^J}}.$$

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$$\sum_{d\le X}^{*}\chi_d(n)=\begin{cases}\frac{X}{2\zeta(2)}a(n)+\text{small}, & \text{if } n=\square,\\ \text{small}, & \text{if } n\neq\square,\end{cases}$$

where $a(n)=\prod_{p|n}\frac{p}{p+1}$.

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where $a(n)=\prod_{p|n}\frac{p}{p+1}$. If $n=\square$, partial summation yields

$$\sum_{d\le X}^{*}\frac{\chi_d(n)}{d^{\beta}}\sim\frac{X^{1-\beta}}{2\zeta(2)(1-\beta)}a(n)$$

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$$\prod_{j\in J} X(s_j) \sum_{n_1,\dots,n_k} \frac{1}{n_1^{s_1^J}\dots n_k^{s_k^J}} \sum_{d\le X}^{*} \frac{\chi_d(n_1\dots n_k)}{d^{\sum\limits_{j\in J} s_j - \frac{|J|}{2}}}.$$

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We thus obtain a term of the form

$$T_J(X; s_1,\dots,s_k) = \frac{X^{1+\frac{|J|}{2}-\sum\limits_{j\in J} s_j}}{2\zeta(2)\left(1+\frac{|J|}{2}-\sum\limits_{j\in J} s_j\right)} \prod_{j\in J} X(s_j) \sum_{n_1\dots n_k=\square} \frac{a(n_1\dots n_k)}{n_1^{s_1^J}\dots n_k^{s_k^J}}.$$

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Recall:

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For $J \subset \{1,\dots,k\}$, we have a term

$$T_J(X;s_1,\dots,s_k) = \frac{X^{1+\frac{|J|}{2}-\sum_{j\in J} s_j}}{2\zeta(2)\left(1+\frac{|J|}{2}-\sum_{j\in J} s_j\right)} \prod_{j\in J} X(s_j) \sum_{n_1\dots n_k=\square} \frac{a(n_1\dots n_k)}{n_1^{s_1^J}\dots n_k^{s_k^J}}.$$

The recipe for real Dirichlet L-functions

Recall:

$$\sum_{d \le X}^{*} L(s_1, \chi_d) \dots L(s_k, \chi_d) \approx \sum_{d \le X}^{*} \prod_{j=1}^{k} \left(\sum_{n \le x} \frac{\chi_d(n)}{n^{s_j}} + X_d(s_j) \sum_{n \le y} \frac{\chi_d(n)}{n^{1-s_j}} \right).$$

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The conjecture is that the moment is the sum of all the 2^k terms:

$$\sum_{d \le X}^{*} L(s_1, \chi_d) \dots L(s_k, \chi_d) = \sum_{J \subset \{1, \dots, k\}} T_J(X; s_1, \dots, s_k) + \text{Error}.$$

Notes

$$T_J(X; s_1, \dots, s_k) = \frac{X^{1+\frac{|J|}{2} - \sum_{j \in J} s_j}}{2\zeta(2)\left(1 + \frac{|J|}{2} - \sum_{j \in J} s_j\right)} \prod_{j \in J} X(s_j) \sum_{n_1 \dots n_k = \square} \frac{a(n_1 \dots n_k)}{n_1^{s_1^J} \dots n_k^{s_k^J}}.$$

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- $|J| = \ell$ are called the ℓ-swap terms.
- In practice, the terms often arise in a different form.

One Dirichlet polynomial

How to recover the result by using just one long Dirichlet polynomial?

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Multiple Dirichlet series

Strategy

Perron's formula gives

$$\sideset{}{^*}\sum_{d\leq X} L(s_1,\chi_d)\dots L(s_k,\chi_d) = \frac{1}{2\pi i}\int_{(2)} A(s_1,\dots,s_k,w)\cdot\frac{X^w}{w}dw,$$

where

$$A(s_1,\dots,s_k,w) = \sideset{}{^*}\sum_{d\geq 1}\frac{L(s_1,\chi_d)\dots L(s_k,\chi_d)}{d^w}.$$

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Strategy:

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Properties of $A(s_1, \ldots, s_k, w)$

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Contribution of the pole at $w = 1$

$$\sum_{d \le X}^{*} L(s_1, \chi_d) \dots L(s_k, \chi_d) = \frac{1}{2\pi i} \int_{(c)} A(s_1, \dots, s_k, w) \cdot \frac{X^w}{w} dw.$$

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The other terms come from functional equations of $A(s_1, \dots, s_k, w)$.

Functional equations

Let $j \in \{1, \ldots, k\}$. We can use the functional equation

$$L(s_j, \chi_d) = d^{1/2-s_j} X(s_j) L(1-s_j, \chi_d):$$

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Let σ_j denote the transformation $(s_1, \dots, s_k, w) \mapsto (s_1, \dots, 1 - s_j, \dots, s_k, w + s_j - 1/2)$. We can also use the functional equation in more variables: for every $J \subset \{1, \dots, k\}$, we obtain a functional equation under $\sigma_J = \prod_{j \in J} \sigma_j$:

$$A(s_1, \dots, s_k, w) = \prod_{j \in J} X(s_j) \cdot A\left(s_1^J, \dots, s_k^J, w + \sum_{j \in J} s_j - \frac{|J|}{2}\right).$$

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This is $T_J(X;s_1,\dots,s_k)$!

Meromorphic continuation

If $A(s_1, \ldots, s_k, w)$ has a meromorphic continuation past the point $(1/2, \ldots, 1/2, 1)$, we can shift the integral to the left, find the contribution of the pole at $w = 1$ and obtain an asymptotic formula with power-saving error term.

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How far can we get the meromorphic continuation?

Main result

Theorem (Č., 2024+)

Assume the Generalized Lindelöf hypothesis, or that $k \leq 4$. Then $A(s_1, \ldots, s_k, w)$ has a meromorphic continuation to the region determined by the half-hyperspaces

$$\begin{aligned} \mathrm{Re}(w) &> 1/2, \\ \mathrm{Re}(s_j + 2w) &> 7/4, \\ \mathrm{Re}(s_{j_1} + s_{j_2} + 2w) &> 5/2, \\ \mathrm{Re}(s_{j_1} + s_{j_2} + s_{j_3} + 2w) &> 13/4, \\ \mathrm{Re}(s_{j_1} + s_{j_2} + s_{j_3} + s_{j_4} + 2w) &> 4. \end{aligned}$$

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- Also applies to long Dirichlet polynomials (recover results of Conrey and Rodgers). The last condition prevents detection of 2-swap terms.

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Assume the generalized Lindelöf hypothesis $|L(\sigma + it, \chi_d)| \ll_t d^\varepsilon$ for $\sigma \geq 1/2$.

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This would recover the Theorem with the condition $\mathrm{Re}(2w + s_1 + s_2) > 3$. How to do better?

Extra functional equation

There is an extra (heuristic) functional equation:

$$A(s_1, \dots, s_k, w) = \sum_{n_1, \dots, n_k} \frac{L_D\left(w, \left(\frac{\cdot}{n_1 \dots n_k}\right)\right)}{n_1^{s_1} \dots n_k^{s_k}}$$

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Equivalent to using Poisson summation.

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Strategy of Diaconu, Goldfeld, Hoffstein and others: Construct a "perfect" MDS

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An extensive theory of Bump, Chinta, Diaconu, Friedberg, Goldfeld, Gunnels, Hoffstein, Whitehead,...

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We work with the unmodified MDS

$$A(s_1, \dots, s_k, w) = \sum_{d \geq 1}^{*} \frac{L(s_1, \chi_d) \dots L(s_k, \chi_d)}{d^w},$$

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A disadvantage is that we can't iterate the functional equations (related to secondary terms).

Functional equation for all characters

Proposition

Let χ be any character modulo n. Then we have

$$L(s,\chi)=\varepsilon(\chi)\left(\frac{\pi}{n}\right)^{s-1/2}\frac{\Gamma\left(\frac{1-s+a}{2}\right)}{\Gamma\left(\frac{s+a}{2}\right)}K(1-s,\chi),$$

where

$$K(s,\chi)=\sum_{k=1}^{\infty}\frac{\tau(\chi,k)}{k^s},\qquad \tau(\chi,q)=\frac{1}{\sqrt{n}}\sum_{j \pmod n}\chi(j)e\left(\frac{kj}{n}\right)$$

is a normalized shifted Gauss sum, and $a=0$ or 1, $\varepsilon(\chi)=1$ or $-i$, depending on the parity of χ.

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If χ is a primitive character, then $\tau(\chi,q) = \bar{\chi}(q)\tau(\chi,1)$, and we recover the usual functional equation.

Proof: follow any proof that uses Poisson summation.

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We thus obtain

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Final region for $A(s_1, \ldots, s_k, w)$

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This can be done and gives the final result.

One-swap terms

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We only used this functional equation to get the meromorphic continuation, not to compute the residues at the poles.

Thank you!