Mixing times and representation theory

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$\ddot{A}$: Motivation: different ways to mix a deck of cards

Example 1: riffle shuffle

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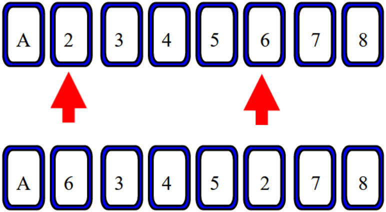
Question: how long does it take to mix?

Ä : The random transposition shuffle

Method :

- Ï Pick two cards uniformly and independently;
- Ï If different, interchange them;
- Ï If they are the same card, do nothing.



Credit: Elchanan Mossel

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Interpretation :

- Random walk on S_n with

$$P(\text{¾},\text{¾¿}) \text{ Æ } \text{¹}_n(\text{¿}) \text{ Æ } \begin{cases} 1/n & \text{if ¿ Æ } id \\ 2/n^2 & \text{if ¿ is a transp.} \end{cases}$$

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Cayley graph for n Æ 3

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Définition

Distance to stationarity after t steps :

$$\mathrm{d}_n(t) :Æ\, \mathrm{d}_{\mathrm{TV}}(^{o}{}_{n}(t), \mathrm{Unif}_n).$$

where for probability measures ^{1} and ^{o} on S_n,

$$\mathrm{d}_{\mathrm{TV}}(^{1}, ^{o})\, Æ \max_{A ½ \mathrm{S}_n} \bar{\bar{\ }}\, ^{1}(A)\ ¡\ ^{o}(A) \bar{\bar{\ }}\, Æ\, \frac{1}{2}\mathrm{d}_1(^{1}, ^{o}).$$

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Question : in which sense do we converge to uniformity?

$\mu_n(t)$: distribution of the walk after t steps.

Définition

Distance to stationarity after t steps :

$$\mathrm{d}_n(t) := \mathrm{d}_{\mathrm{TV}}(\mu_n(t), \mathrm{Unif}_n).$$

where for probability measures ν and μ on $\mathfrak{S}_n$,

$$\mathrm{d}_{\mathrm{TV}}(\nu, \mu) = \max_{A \subset \mathfrak{S}_n} \left| \nu(A) - \mu(A) \right| = \frac{1}{2} \mathrm{d}_1(\nu, \mu).$$

Question : How large should be take t so that $d_n(t) \approx 0$?

Ä : Cutoff for random transpositions

Theorem (Diaconis and Shahshahani, 1981)

It takes $\frac{1}{2}n\ln(n)$ steps to mix a deck of n cards by random transpositions. For every $0 \leq \epsilon \leq 1$,

$$d_n\left((1-\epsilon)\frac{1}{2}n\ln(n)\right) \xrightarrow[n\to+\infty]{} 1 \quad \& \quad d_n\left((1+\epsilon)\frac{1}{2}n\ln(n)\right) \xrightarrow[n\to+\infty]{} 0$$

That is what is called the **cutoff phenomenon**.

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That is what is called the **cutoff phenomenon**.
More precisely, it takes $\frac{1}{2}n\ln(n) \pm \Theta(n)$ steps to mix.

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New York Times, 9 Jan. 1990

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Answer: Bayer–Diaconis 1992: Precise estimates for $n = 52$, and cutoff profile:

Theorem (Bayer–Diaconis, 1992)

For the riffle shuffle, we have for every $c \in \mathbb{R}$,

$$d_n\left(\frac{3}{2}\log_2(n) + c\right) \xrightarrow[n\to\infty]{} p(c) := \mathrm{d}_{\mathrm{TV}}\left(\mathcal{N}(0,1), \mathcal{N}\left(\frac{2^{-c}}{2\sqrt{3}}, 1\right)\right).$$

(Written up to integer parts.)

$\ddot{A}$: Cutoff profile for random transpositions

Question: and for transpositions, can we find the profile?
Question asked by N. Berestycki at an AIM workshop in 2016.

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Theorem (T., 2020)

For random transpositions, we have for every $c \in \mathbb{R}$,

$$d_n\left(\frac{1}{2}n\ln(n) + cn\right) \xrightarrow[n\to+\infty]{} p(c) := d_{\mathrm{TV}}\left(\mathrm{Poiss}\left(1+e^{-2c}\right), \mathrm{Poiss}(1)\right).$$

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- Several different types of profiles are known. For example with
  - **normal laws** for the riffle shuffle (Bayer–Diaconis, 1992), the random walk on $(\mathbb{Z}/2\mathbb{Z})^n$ (Diaconis–Graham–Morrison, 1990)), or simple excusion process on the circle (Lacoin 2016),
  - **Poisson laws** for k-cycles ($k = o(n)$, Nestoridi–Olesker-Taylor, 2022) or more generally all congugacy classes of the symmetric group (Olesker–Taylor T. 2024?),
  - **Tracy–Widom distributions** for the ASEP on a segment (Bufetov–Nejjar 2022),
  - **free Meixner laws** for the diffusion on O_N^+ (Freslon–T.–Wang, 2022).

$\ddot{A}$: Some results related to random transpositions

On random transpositions themselves :

Cutoff result : Diaconis, Shahshahani, 1981, *PTRF*
Precise lower bound : Matthews, 1988, *J. of Th. Prob.*
Phase transition result : N. Berestycki, Durrett, 2006, *PTRF*
More precise estimates on the cutoff window : Saloff-Coste–Zuniga, 2010, *AAP*
Probability of long cycles : Alon, Kozma, 2013, *Duke*
Strong stationary time : White, 2019
Cutoff profile : T., 2020, *Ann. Prob.*

Generalisations to other conjugacy classes :

Almost-precutoff for all conjugacy classes Roichman, 1996, *Invent. Math.*
Some conjugacy classes with few fixed points Lulov–Pak, 2002, *J. Alg. Comb.*
Precutoff for all conjugacy classes with few fixed points Larsen–Shalev, 2008, *Invent. Math.*
Cutoff for k-cycles : N. Berestycki, Schramm, Zeitouni, 2011, *Ann. Prob.*
Cutoff for conjugacy-invariant walks on $\mathfrak{S}_n$: N. Berestycki, Şengül, 2014, *PTRF*
Profile for k-cycles : Nestoridi, Olesker-Taylor, 2021, *PTRF*
Cutoff + profile for all conjugacy classes : Olesker-Taylor–T., 2024?

Some other generalisations :

Biaised random transpositions : Matheau-Raven, 2020
Quantum random transpositions : Freslon, T., Wang, 2021, *PTRF*
Star random transpositions : Nestoridi, 2021

~ : The non-commutative Fourier transform

Using the **Fourier transform** : **key point** to study the walk.
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but instead the one of **finite groups** G, where for $\rho \in \hat{G}$,

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Inverse Fourier transform, isometry between Hilbert spaces, Parseval identity.

Pierre-Loïc Méliot, *Representation Theory of Symmetric Groups*, chap. 1.

~ : A method to find cutoff profiles

For transpositions, we then apply the **inverse Fourier transform** on S_n to $f := \pi_n(t) - \mathrm{Unif}_n$, and use that μ_n is **constant on conjugacy classes** (so by Schur's lemma each $\hat{f}(\lambda)$ is a multiple of the identity (as a matrix)), to get

$$2d_n(t) = \frac{1}{n!} \sum_{\sigma \in S_n} \left| \sum_{\lambda \in \widehat{S}_n^*} d_\lambda s_\lambda^t \chi_\lambda(\sigma) \right|.$$

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$$2\mathrm{d}_n(t) Æ \frac{1}{n!} \sum_{¾2 S_n} \left| \sum_{¸ 2 \hat{S}_n} \mathrm{d}_¸ s_¸^t \mathrm{ch}_¸(¾) \right|.$$

$s_¸$: eigenvalues of the (transition matrix of the) chain

d$_¸$: multiplicities

ch$_¸$(¾): "eigenvectors".

~ : Representations of the symmetric group

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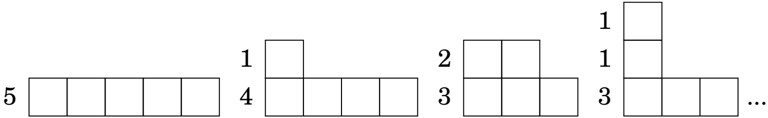
  (i.e. partitions ¸ Æ (¸$_1$, ¸$_2$,...) of the integer *n*.)
  Example: *n* Æ 5 Æ 4 Å 1 Æ 3 Å 2 Æ ....



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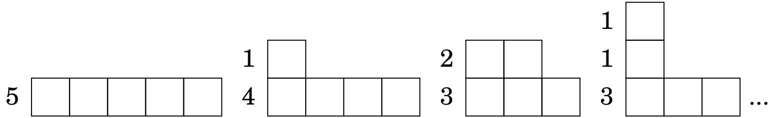
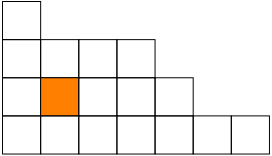
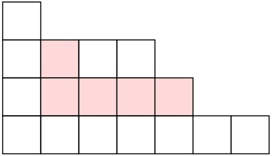
- Irreducible representations λ of S_n $\leftrightarrow$ Young diagrams of size n. (i.e. partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of the integer n.)
  Example: $n = 5 = 4 + 1 = 3 + 2 = \ldots$.



- **Hook-length (of a box):** Example: $\lambda = [7,5,4,1]$, u = orange box. The hook-length of u is $H(\lambda, u) = 5$.





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The dimensions d_λ are easy to compute:

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Example

The diagram $\lambda = [3,2]$ has 5 standard tableaux, so $d_\lambda = |\mathrm{ST}(\lambda)| = 5$.

4	5	
1	2	3

3	5	
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Example

The hook-lengths of (the boxes of) $\lambda = [3,2]$ are 4, 3, 1, 2, 1, so
$d_\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} H(\lambda,u)} = \frac{5!}{4\cdot 3\cdot 1\cdot 2\cdot 1} = 5.$

2	1	
4	3	1

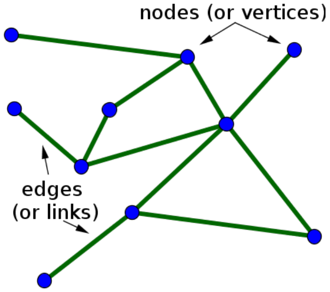
.

} : Another type of problems: the cover time problem

¡ Æ (V, E): graph (finite, connected).

$(X_t)_{t\,¸\,0}$: simple random walk on ¡.

T_x = "hitting time of the vertex x" = $\min\{t : X_t$ Æ $x\}$.



Credits: mathinsight.org

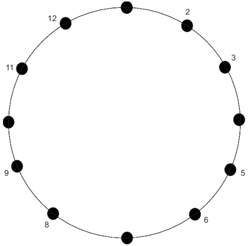
Cover time: ¿$_{\text{cov}}$ = $\max_{x2V} T_x$ = "first time at which all vertices have been visited".

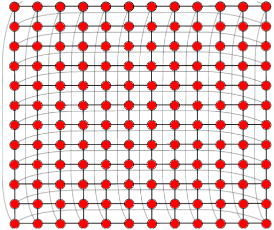
Research on cover times: active since the works of Aldous in the 80's.

Question: E[¿$_{\text{cov}}$]? Is ¿$_{\text{cov}}$ concentrated?

} : Cover time of tori

Example: the d-dimensional torus $(\mathbb{Z}/m\mathbb{Z})^d$. $n := m^d$.





Credits: Ljupco Kocarev ($d = 1$), Markus Quade ($d = 2$), researchgate

- $d = 1$: $\tau_{\text{cov}} \asymp n^2$, not concentrated.
- $d = 2$: cover time cutoff: $\tau_{\text{cov}} \sim c_2 n (\log n)^2$
  (Dembo–Peres–Rosen–Zeitouni 2004, *Ann. Math.*)
- $d \geq 3$, cover time profile: $\tau_{\text{cov}} \approx c_d n(\log n + G)$, where $G \sim$ Gumbel, i.e. $\mathbb{P}(G \leq s) = e^{-e^{-s}}$.
  (Belius 2013, *Ann. Prob.* / De Prata 2012)

Our result: characterisation of Gumbel fluctuations.

$t_{\text{hit}} = t_{\text{hit}}(\Gamma) = \max_{x,y\in V} \mathbb{E}_x T_y$.

$\text{Diam}(\Gamma)$ = diameter of $\Gamma = \max_{x,y\in\Gamma} d(x,y)$.

$n = n(\Gamma) := |V|$.

Theorem (N. Berestycki–Hermon–T. 2023+)

For vertex-transitive graphs of (uniformly) bouned degree, we have

$$\frac{\tau_{\text{cov}}}{t_{\text{hit}}} - \log n \xrightarrow[n\to\infty]{} \Lambda \quad \textit{if and only if} \quad \text{Diam}(\Gamma)^2 \log n = o(n).$$

} : Cover time of vertex-transitive graphs

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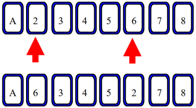
- "Gumbel iff Γ is a bit more than *2-dimensional*"
- Also iff the last points to be covered are "uniform"

} : Thank you for your attention!

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