Joint distribution of primes in multiple short intervals

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Outline

Section 1

Introduction

Origin

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In his famous letter to Encke, Gauss counted primes in intervals of length 100 from 1 million to 3 million, and also compared $\pi(n+H) - \pi(n)$ and $\mathrm{Li}(n+H) - \mathrm{Li}(n)$ with $H = 10^5$.

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Chiliades.

1	168	51	89	101	81	151	85	201	77	251	71	301	85	351	74	401	70	451	92
2	135	52	97	102	93	152	90	202	87	252	88	302	83	352	80	402	71	452	76
3	127	53	89	103	87	153	88	203	78	253	78	303	72	353	82	403	76	453	69
4	120	54	92	104	80	154	77	204	78	254	81	304	84	354	76	404	75	454	72
5	119	55	90	105	91	155	84	205	77	255	76	305	88	355	87	405	70	455	74
6	114	56	93	106	82	156	85	206	85	256	87	306	80	356	79	406	83	456	82
7	117	57	99	107	92	157	76	207	83	257	72	307	82	357	67	407	67	457	73
8	107	58	91	108	76	158	88	208	87	258	78	308	73	358	80	408	81	458	77
9	110	59	90	109	91	159	87	209	85	259	86	309	76	359	83	409	79	459	75
10	112	60	94	110	88	160	85	210	88	260	76	310	80	360	71	410	82	460	68
11	106	61	88	111	83	161	85	211	84	261	77	311	79	361	68	411	73	461	77
12	103	62	87	112	84	162	84	212	86	262	73	312	69	362	79	412	81	462	69
13	109	63	88	113	81	163	81	213	69	263	79	313	86	363	76	413	74	463	74
14	105	64	93	114	88	164	83	214	81	264	84	314	86	364	84	414	69	464	77
15	102	65	80	115	82	165	77	215	86	265	80	315	76	365	77	415	90	465	85
16	108	66	98	116	93	166	80	216	74	266	78	316	77	366	77	416	80	466	74
17	98	67	84	117	81	167	81	217	76	267	87	317	84	367	85	417	67	467	69
18	104	68	99	118	90	168	83	218	80	268	94	318	84	368	79	418	82	468	83
19	94	69	80	119	79	169	73	219	84	269	75	319	81	369	72	419	85	469	85
20	102	70	81	120	87	170	87	220	91	270	78	320	86	370	68	420	75	470	72
21	98	71	98	121	88	171	87	221	78	271	84	321	79	371	70	421	75	471	87
22	104	72	95	122	86	172	81	222	80	272	78	322	80	372	76	422	73	472	78
23	100	73	90	123	88	173	89	223	81	273	83	323	81	373	81	423	77	473	73
24	104	74	83	124	88	174	79	224	80	274	71	324	71	374	73	424	83	474	78
25	94	75	92	125	83	175	83	225	83	275	80	325	87	375	82	425	81	475	80
26	98	76	91	126	84	176	75	226	84	276	83	326	85	376	85	426	74	476	86
27	101	77	83	127	83	177	95	227	76	277	83	327	73	377	80	427	71	477	75
28	94	78	95	128	86	178	73	228	80	278	74	328	86	378	71	428	78	478	69
29	98	79	84	129	89	179	89	229	89	279	81	329	73	379	77	429	71	479	85
30	92	80	91	130	83	180	94	230	88	280	73	330	81	380	83	430	89	480	71
31	95	81	88	131	85	181	71	231	84	281	87	331	80	381	72	431	76	481	77
32	92	82	92	132	83	182	79	232	78	282	85	332	82	382	76	432	79	482	78
33	106	83	89	133	87	183	91	233	76	283	77	333	72	383	74	433	84	483	82
34	100	84	84	134	82	184	79	234	71	284	72	334	80	384	81	434	80	484	75
35	94	85	87	135	80	185	83	235	87	285	90	335	77	385	78	435	85	485	65
36	92	86	85	136	89	186	91	236	73	286	77	336	77	386	80	436	82	486	63
37	99	87	88	137	96	187	79	237	76	287	71	337	84	387	78	437	73	487	82
38	94	88	93	138	80	188	87	238	73	288	71	338	80	388	69	438	70	488	78
39	90	89	76	139	85	189	80	239	87	289	85	339	77	389	75	439	75	489	83
40	96	90	94	140	84	190	88	240	79	290	84	340	68	390	84	440	75	490	78
41	88	91	89	141	87	191	75	241	80	291	84	341	84	391	81	441	79	491	78
42	101	92	85	142	87	192	83	242	91	292	77	342	77	392	79	442	72	492	76
43	102	93	97	143	82	193	89	243	76	293	78	343	77	393	86	443	85	493	67
44	85	94	86	144	77	194	84	244	77	294	68	344	80	394	87	444	88	494	82
45	96	95	87	145	79	195	74	245	78	295	85	345	80	395	75	445	82	495	80
46	86	96	95	146	85	196	85	246	80	296	75	346	76	396	72	446	68	496	87
47	90	97	84	147	84	197	76	247	84	297	82	347	80	397	75	447	68	497	68
48	95	98	82	148	83	198	87	248	79	298	73	348	82	398	75	448	73	498	81
49	89	99	87	149	83	199	96	249	88	299	73	349	77	397	82	449	70	499	72
50	98	100	87	150	91	200	77	250	80	300	78	350	82	400	81	450	80	500	81

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Primzahlen

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1.	1.										1.
2.		1.				1.		1.	1.		4.
3.		4.	2.	2.	3.	1.	2.	3.	3.	1.	21.
4.	2.	8.	5.	4.	3.	6	9.	4.	5.	8.	54.
5	11.	10.	8.	18.	12.	10.	10.	12.	15.	8	114
6	14.	14.	18.	21.	16.	22.	19.	15.	17.	15.	171.
7.	26	17.	23.	23.	24.	24.	17.	22.	20.	21.	217.
8.	19.	19.	21.	7.	14.	15.	20.	17.	15.	17.	164.
9.	11.	13.	9.	13.	14.	14.	12.	13.	11.	16.	126.
10.	8.	6.	8.	5.	9.	5.	5.	9.	7.	9.	71.
11.	6.	6.	4.	6.	3.	1.	3.	1.	4.	5.	39.
12.	1.	1.	2.	1.	1.	1.	2.	2.	1.		12.
13.	1.	1.			1.		1.	1.	1.		6.
14											
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	752	719	732.	700.	731.	698.	713.	722.	706.	737.	7210.

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Mean and variance

Applying the prime number theorem, the mean of $\psi(n+H)-\psi(n)$ for $n \in [1, N]$ is

$$\frac{1}{N}\sum_{n \leqslant N}(\psi(n+H)-\psi(n)) \sim H,$$

provided that $H = o(N)$ as $N \to \infty$.

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In 1973, Goldston and Montgomery showed that the variance of $\psi(n+H)-\psi(n)$ for $n \in [1, N]$ is $\sim H \log \frac{N}{H}$ in the range of $H \in \left[N^{\epsilon}, N^{1-\epsilon}\right]$ under the Riemann hypothesis (RH) and the strong pair correlation conjecture.

Higher moments and normality

In 2004, by computing higher moments under a uniform Hardy–Littlewood prime k-tuple conjecture, Montgomery and Soundararajan showed that the distribution of $\psi(n+H)-\psi(n)$ for $n \in [1, N]$ is approximately normal with mean $\sim H$ and variance $\sim H \log \frac{N}{H}$, provided that $\frac{H}{\log N} \to \infty$ and $\frac{\log H}{\log N} \to 0$ as $N \to \infty$.

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They further conjectured that in the range of $H \in \left[(\log N)^{1+\delta}, N^{1-\delta}\right]$, the distribution remains to be normal.

Rubinstein–Sarnak approach

In this paper, we revisit the "Fourier side". However, rather than relying on the analytic nature of the pair correlation conjecture, we adapt the method of Rubinstein and Sarnak and assume the linear independence over $\mathbb{Q}$ of the positive ordinates of nontrivial zeros (LI), which is an algebraic assumption.

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Given a large X and $x \in [2, X]$. As we will see, for $h = h(x) = \delta x$, where $\delta > 0$ is small but independent of X, which is beyond the conjectural range of Montgomery and Soundararajan above, the distribution of $\psi(x+h) - \psi(x)$ for $x \in [2, X]$ (in logarithmic scale) remains to be Gaussian under RH and LI.

Primes in two neighbouring intervals

Furthermore, one may ask: What is the joint distribution of the weighted prime counts in two neighbouring intervals? Are they independent? If not, how are they correlated?

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In fact, we show that assuming RH and LI, the weighted count $(\psi(x) - \psi(x-h), \psi(x+h) - \psi(x))$ for $x \in [2, X]$ (in logarithmic scale) is approximately bivariate Gaussian with a weak negative correlation.

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More generally, we show that the weighted count of primes in multiple disjoint short intervals has a multivariate Gaussian (logarithmic) limiting distribution with a weak negative correlation under RH and LI.

In the case of two neighbouring intervals, we have:

Corollary

Assume RH and LI. Given a Borel subset $B \subseteq \mathbb{R}^2$, *define*

$$S_{X,\delta;B} := \left\{ x \in [2, X] \, : \, \frac{(\psi(x) - \psi(x - \delta x) - \delta x, \psi(x + \delta x) - \psi(x) - \delta x)}{\sqrt{\left(\delta \log \frac{1}{\delta} + (1 - \gamma - \log 2\pi)\delta\right) x}} \in B \right\}.$$

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Then as $\delta \to 0^+$, we have

$$\lim_{X \to \infty} \frac{1}{\log X} \int_{S_{X,\delta;B}} \frac{dx}{x} = \frac{1}{2\pi\sqrt{\det \mathcal{C}}} \int_B \exp\left(-\frac{1}{2}\langle \mathcal{C}^{-1}\boldsymbol{x}, \boldsymbol{x}\rangle\right) d\boldsymbol{x} + O\left(\frac{1}{\log^2 \frac{1}{\delta}}\right)$$

with the covariance matrix

$$\mathcal{C} = \begin{pmatrix} 1 & -\frac{\log 2}{\log \frac{1}{\delta}} \\ -\frac{\log 2}{\log \frac{1}{\delta}} & 1 \end{pmatrix}.$$

Section 2

Preliminaries

Normalized deviation

Given an integer $r \geqslant 1$, real numbers $x \geqslant 2, \delta > 0$ and a vector $\boldsymbol{t} \in \mathbb{R}^r$, we denote by $\boldsymbol{E}(x; \delta, \boldsymbol{t})$ the r-tuple $(E(x; \delta, t_1), \ldots, E(x; \delta, t_r))$, where

$$E(x; \delta, t) := \frac{1}{\sqrt{x}} \left[\psi\left((1 + t\delta)x + \frac{1}{2}\delta x \right) - \psi\left((1 + t\delta)x - \frac{1}{2}\delta x \right) - \delta x \right],$$

i.e. the normalized deviation of the weighted prime count in the short interval of length δx centred at $(1 + t\delta)x$.

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i.e. the normalized deviation of the weighted prime count in the short interval of length δx centred at $(1+t\delta)x$.

To simplify our discussion, we always assume that

- $T := \max_{j=1,\ldots,r}|t_j| \leqslant \delta^{-\frac{1}{10}}$, i.e. the intervals are not too spread out;
- $|t_j - t_k| \geqslant 1$ whenever $j \neq k$, i.e. the intervals are disjoint.

Logarithmic limiting distribution

Let $\boldsymbol{Y}(x)$ be a $\mathbb{R}^r$-valued function. We say that $\boldsymbol{Y}(x)$ has a logarithmic limiting distribution μ on $\mathbb{R}^r$ if

$$\begin{aligned}\mathbb{E}_x^{\log}\left[f\left(\boldsymbol{Y}(x)\right)\right] &:= \lim_{X\to\infty}\frac{1}{\log X}\int_2^X f(\boldsymbol{Y}(x))\frac{dx}{x} \\ &= \int_{\mathbb{R}^r} f(\boldsymbol{y})d\mu(\boldsymbol{y})\end{aligned}$$

for all bounded continuous functions f on $\mathbb{R}^r$, i.e. (logarithmic) time average equals space average.

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for all bounded continuous functions f on $\mathbb{R}^r$, i.e. (logarithmic) time average equals space average.

If such a measure μ exists, then given a Borel subset $B \subseteq \mathbb{R}^r$, we shall represent $\mu(B)$ by

$$\mathbb{P}_x^{\log}\left(\boldsymbol{Y}(x)\in B\right) := \mathbb{E}_x^{\log}\left[1_B\left(\boldsymbol{Y}(x)\right)\right],$$

where $1_B(\boldsymbol{x})$ is the indicator function of the Borel subset B.

Given a non-trivial zero $\rho = \frac{1}{2} + i\gamma$, we define

$$w(\rho) = w(\rho; \delta, t) := \frac{1}{s}\left[\left(1 + \left(t + \frac{1}{2}\right)\delta\right)^{\rho} - \left(1 + \left(t - \frac{1}{2}\right)\delta\right)^{\rho}\right].$$

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Proposition

Let $r \geqslant 1, \delta > 0$ and $\boldsymbol{t} \in \mathbb{R}^r$ be fixed. Assume RH and LI. Then $\boldsymbol{E}(x; \delta, \boldsymbol{t})$ has a logarithmic limiting distribution $\mu_{\delta,\boldsymbol{t}}$ on $\mathbb{R}^r$ corresponding to the $\mathbb{R}^r$-valued random vector $\boldsymbol{X}_{\delta,\boldsymbol{t}} = (X_{\delta,t_1}, \ldots, X_{\delta,t_r})$, where

$$X_{\delta,t} := \mathrm{Re}\left(2\sum_{\gamma>0} w(\rho)U_\gamma\right)$$

with $\{U_\gamma\}_{\gamma>0}$ being a sequence of independent random variables uniformly distributed on the unit circle $\mathbb{T}$.

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with $\{U_\gamma\}_{\gamma>0}$ being a sequence of independent random variables uniformly distributed on the unit circle $\mathbb{T}$. Moreover, the covariance matrix of $\boldsymbol{X}_{\delta,\boldsymbol{t}}$ is real symmetric with the (j,k)-entry being

$$\mathrm{Cov}_{jk} = \mathrm{Cov}_{jk}(\delta, \boldsymbol{t}) := \sum_{\gamma} w_j(\rho)\overline{w_k(\rho)}.$$

Covariance

Recall that

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Proposition

Assume RH and LI. Then as $\delta \to 0^+$, we have

$$\mathrm{Cov}_{jk} = \begin{cases} \delta \log \frac{1}{\delta} + (1 - \gamma - \log 2\pi)\delta + O\left(\left(T\delta \log \frac{1}{\delta}\right)^2\right) & \text{if } j = k, \\ -\Delta(|t_j - t_k|)\delta + O\left(\left(T\delta \log \frac{1}{\delta}\right)^2\right) & \text{if } j \neq k, \end{cases}$$

where

$$\Delta(t) := \frac{1}{2}\left((t+1)\log(t+1) - 2t\log t + (t-1)\log(t-1)\right),$$

i.e. the second order central difference of the function $f(t) = \frac{1}{2} t \log t$.

The appearance of the secondary term $(1-\gamma-\log 2\pi)\delta$ is expected as the variance computed by Montgomery and Soundararajan is

$$\frac{1}{X}\int_1^X (\psi(x+H)-\psi(x)-H)^2\, dx \sim H\log\frac{X}{H}-(\gamma+\log 2\pi)H$$

for $X^\epsilon \leqslant H \leqslant X^{1-\epsilon}$ under a uniform Hardy–Littlewood prime k-tuple conjecture.

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Note that $\Delta(1)=\log 2$ and $\Delta(|t_j-t_k|)>0$ in general. Also, as $|t_j-t_k|\to\infty$, we have $\Delta(|t_j-t_k|)\to 0^+$ monotonically and more precisely, $\Delta(|t_j-t_k|)\sim\frac{1}{2|t_j-t_k|}$, i.e. "Coulomb's law".

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Therefore, primes in disjoint short intervals repelled each other, albeit very weakly. Moreover, the larger the gap between the intervals, the weaker the repulsion.

Section 3

Main results

Statement

In view of our first proposition, we shall state our main theorems in terms of the renormalized deviation

$$\widetilde{\boldsymbol{E}}(x;\delta,\boldsymbol{t}) := \left(\frac{E(x;\delta,t_1)}{\sqrt{V_1}},\ldots,\frac{E(x;\delta,t_r)}{\sqrt{V_r}}\right),$$

where $V_j := \mathrm{Cov}_{jj}$ for $j = 1,\ldots,r$.

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where $V_j := \mathrm{Cov}_{jj}$ for $j = 1,\dots,r$.

Theorem

Assume RH and LI. Given a small $\delta > 0$ and an integer $1 \leqslant r \leqslant \frac{\log 1/\delta}{\log\log 1/\delta}$. Let $B \subseteq \mathbb{R}^r$ be a Borel subset. Then as $\delta \to 0^+$, the total variation distance

$$\sup_{B\subseteq\mathbb{R}^r\,:\,B\text{ Borel}} \left|\mathbb{P}_x^{\log}\left(\widetilde{\boldsymbol{E}}(x;\delta,\boldsymbol{t})\in B\right) - \mathbb{P}(\mathcal{N}(\mathbf{0},\mathcal{C})\in B)\right| \ll_{r,T} \delta\left(\log\frac{1}{\delta}\right)^{\frac{r}{2}-1},$$

where $\mathcal{N}(\mathbf{0},\mathcal{C})$ is an r-dimensional Gaussian random variable with mean $\mathbf{0}$ and covariance matrix $\mathcal{C} = (c_{jk})_{1\leqslant j,k\leqslant r}$ and

$$c_{jk} = c_{jk}(\delta,\boldsymbol{t}) := \frac{\mathrm{Cov}_{jk}}{\sqrt{V_jV_k}} = -\frac{\Delta(|t_j-t_k|)}{\log\frac{1}{\delta}} + O_{r,T}\left(\frac{1}{\log^2\frac{1}{\delta}}\right)$$

for $1 \leqslant j,k \leqslant r$.

Shanks–Rényi prime number race

In 1853, Chebyshev noted that on a fine scale there seem to be more primes congruent to 3 than to 1 modulo 4, which is now known as the Chebyshev's bias. This observation led to the birth of comparative prime number theory, which investigates the discrepancies in the distribution of prime numbers.

Shanks–Rényi prime number race

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A central problem is the so-called "Shanks–Rényi prime number race. Let $q \geqslant 3$ and $2 \leqslant r \leqslant \varphi(q)$ be positive integers, and denote by $\mathcal{A}_r(q)$ the set of ordered r-tuples $(a_1, \ldots, a_r)$ of distinct residue classes that are coprime to q. Is it true that for any $(a_1, \ldots, a_r) \in \mathcal{A}_r(q)$, we will have the ordering

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Assuming GRH and GLI, Rubinstein and Sarnak showed that this has a positive (logarithmic) density, denoted by $\delta(q; a_1, \ldots, a_r)$.

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Theorem (Harper–Lamzouri, 2018; Ford–Harper–Lamzouri, 2019)

Let $\epsilon > 0$ be small and q be sufficiently large.

- *(Uniformly for small r) If $r \leqslant \log q/(\log\log q)^4$, then uniformly for all n-tuples $(a_1, \ldots, a_r) \in \mathcal{A}_r(q)$, we have $\delta(q; a_1, \ldots, a_r) \sim 1/r!$ as $q \to \infty$.*
- *(Biases for large r) If $r/\log q \to \infty$ as $q \to \infty$, then there exists n-tuples $(a_1, \ldots, a_r), (b_1, \ldots, b_r) \in \mathcal{A}_r(q)$ for which $r! \cdot \delta(q; a_1, \ldots, a_r) \to 0$ and $r! \cdot \delta(q; b_1, \ldots, b_r) \to \infty$.*

Many intervals

To simply notation, let us denote

$$\rho(\delta; \boldsymbol{t}) := \mathbb{P}_x^{\log}\left(\widetilde{E}(x;\delta,t_1) > \widetilde{E}(x;\delta,t_2) > \cdots > \widetilde{E}(x;\delta,t_r)\right).$$

When the number of intervals r is not necessarily bounded as $\delta \to 0^+$, we have the following short-interval analog of a result by Harper–Lamzouri.

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When the number of intervals r is not necessarily bounded as $\delta \to 0^+$, we have the following short-interval analog of a result by Harper–Lamzouri.

Corollary

Assume RH and LI. Given a small $\delta > 0$ and an integer $1 \leqslant r \leqslant \frac{\log 1/\delta}{\log\log 1/\delta}$. Then as $\delta \to 0^+$, we have

$$\rho(\delta; \boldsymbol{t}) = \frac{1}{r!}\left(1 + O\left(\frac{r \log 2r}{\log \frac{1}{\delta}}\right)\right),$$

i.e. all r-way prime number races remain asymptotically unbiased as long as $r = o\left(\frac{\log 1/\delta}{\log\log 1/\delta}\right)$.

Sharp phase transition

When $r \asymp \frac{\log 1/\delta}{\log\log 1/\delta}$, however, it turns out there exist r intervals such that the corresponding prime number race is noticeably biased.

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$$\rho(\delta; \boldsymbol{t}) \leqslant \exp\left(-\eta_0 \times \frac{r \log\log \frac{1}{\delta}}{\log \frac{1}{\delta}}\right) \frac{1}{r!}.$$

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Problem. When $r \asymp \frac{\log 1/\delta}{\log\log 1/\delta}$, is it possible to generate a positive bias, i.e.

$$\rho(\delta; \boldsymbol{t'}) \geqslant \exp\left(+\eta_0' \times \frac{r \log\log \frac{1}{\delta}}{\log \frac{1}{\delta}}\right) \frac{1}{r!} \quad ?$$

Informal Conclusion. Weighted prime counts in multiple short intervals behave as if they are jointly normally distributed point charges.

Thank you for your attention!