A Discrete Mean Value of the Riemann Zeta Function and its Derivatives

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Part I: The Riemann Zeta Function

Preliminaries

Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \qquad (\sigma > 1)$$

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and Riemann proved that $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s)$$

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A zero of $\zeta(s)$ in the critical strip is called a **nontrivial** zero and it is denoted by $\rho = \beta + i\gamma$ where $0 \leqslant \beta \leqslant 1$ and $\gamma \in \mathbb{R}$.

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$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + O\left(\log T\right)$$

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But why are we interested in the zeros of $\zeta(s)$? This is simply because we would like to divide by $\zeta(s)$ in order to have a better understanding on the distribution of prime numbers. To make division by $\zeta(s)$ meaningful, we need to know that $\zeta(s) \neq 0$.

Preliminaries

More precisely, for $\sigma = \Re(s) > 1$, we have

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which gives

$$-\log \zeta(s) = \sum_p \log \left(1 - \frac{1}{p^s}\right)$$

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and by differentiation with respect to s, we have

$$\begin{aligned}
-\frac{\zeta'}{\zeta}(s) = \sum_p \frac{p^{-s}\log p}{1-\frac{1}{p^s}} &= \sum_p p^{-s}\log p\left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) \\
&= \sum_{j\geqslant 1}\sum_p \frac{\log p}{p^{js}}.
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$$= \sum_{j\geqslant 1}\sum_p \frac{\log p}{p^{js}}.$$

For a natural number n, let $\Lambda(n)$ be the von Mangoldt function defined by $\Lambda(n) = \log p$ if n is a prime power p^j for some $j \geqslant 1$, and $\Lambda(n) = 0$ otherwise. Then

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad (\sigma > 1).$$

Thus the function $-\frac{\zeta'}{\zeta}(s)$ is closely related to prime powers. For $x \geqslant 2$, by the Residue Theorem, we have

$$\frac{1}{2\pi i}\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\frac{x^s}{s}\,ds = \sum_{n=1}^{\infty}\Lambda(n)\frac{1}{2\pi i}\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty}\frac{\left(\frac{x}{n}\right)^s}{s}\,ds = \sum_{n\leqslant x}\Lambda(n) + O(\log x)$$

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and

$$\frac{1}{2\pi i}\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty}\left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}\,ds = x - \sum_{\rho}\frac{x^{\rho}}{\rho} + O\left(1\right)$$

and thus

$$\sum_{n\leqslant x}\Lambda(n) = x - \sum_{\rho}\frac{x^{\rho}}{\rho} + O(\log x).$$

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The Prime Number Theorem, conjectured by Gauss in 1792 and proved by Hadamard and de la Vallée Poussin in 1896 independently, is the statement that

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or equivalently that

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Thus the real parts of the nontrivial zeros ρ of $\zeta(s)$ play an important role to control the error term in the Prime Number Theorem.

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The best known zero-free region for the Riemann zeta function is due to Korobov and Vinogradov independently in 1958 that $\zeta(s) \neq 0$ in the region

$$\sigma > 1 - \frac{C}{(\log |t|)^{2/3} (\log \log |t|)^{1/3}}$$

for some positive constant C and $|t| \geqslant 3$ whereas the Riemann Hypothesis is the conjecture that $\zeta(s) \neq 0$ if $\sigma > \frac{1}{2}$.

Part II: Main Results

Discrete Moments of the Riemann Zeta Function

The discrete $2k^{\text{th}}$ moment of the m^{th} derivative of the Riemann zeta function is the sum

$$\sum_{\substack{\rho \\ 0<\gamma\leqslant T}} \left|\zeta^{(m)}(\rho)\right|^{2k}, \quad (k \in \mathbb{R},\ m \in \mathbb{N}).$$

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One of the reasons to study this object is to have a better understanding on the average size of the derivatives of $\zeta(s)$ at its zeros. Moreover, such moments can produce results on the large or small gaps between the ordinates of the zeros of $\zeta(s)$, and multiplicities of the zeros, and the summatory function of the Möbius function.

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Assuming the Riemann Hypothesis, Gonek proved that

$$\sum_{\substack{\rho \\ 0<\gamma\leqslant T}} \left|\zeta'(\rho)\right|^{2} \sim \frac{T}{24\pi} \log^4 T$$

and no other asymptotic formula is known even conditionally.

Discrete Moments of the Riemann Zeta Function

Gonek and Hejhal independently conjectured that

$$T\left(\log T\right)^{k(k+2)+1} \ll \sum_{0<\gamma\leqslant T} \left|\zeta'(\rho)\right|^{2k} \ll T\left(\log T\right)^{k(k+2)+1}$$

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As an upper bound, Kirila proved, under the assumption of the Riemann Hypothesis, that

$$\sum_{0<\gamma\leqslant T} \left|\zeta^{(m)}(\rho)\right|^{2k} \ll T\,(\log T)^{k(k+2m)+1}$$

for $k > 0$ and $m \in \mathbb{N}$.

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Milinovich and Ng proved under the assumption of the Generalized Riemann Hypothesis that

$$\sum_{0<\gamma\leqslant T} \left|\zeta'(\rho)\right|^{2k} \gg T(\log T)^{k(k+2)+1} \tag{1}$$

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for $k \in \mathbb{N}$. Very recently, Heap, Li and Zhao obtained the same lower bound in (1) for rational $k \leqslant 0$ assuming the Riemann Hypothesis and the simplicity of the zeros.

Main Results

Theorem (Kübra Benli, E. , Nathan Ng)

Assume the Riemann Hypothesis. Let $k, m \geqslant 1$ be natural numbers. We have

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The result above is obtained by a general result on the sum

$$S(\alpha, T, X, Y) := \sum_{0<\gamma\leqslant T} \zeta(\rho+\alpha)X(\rho)Y(1-\rho)$$

where $X(s)$ and $Y(s)$ are some Dirichlet polynomials and the shift $\alpha \in \mathbb{C}$ satisfies $|\alpha| \ll \frac{1}{\log T}$.

Definitions and Assumptions

Let

$$X(s) = \sum_{n \leqslant N} \frac{x(n)}{n^s},$$
$$Y(s) = \sum_{n \leqslant N} \frac{y(n)}{n^s}$$

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Assume that $N \ll T^{\vartheta}$ for some $0 < \vartheta < \frac{1}{2}$. Assume further that the submultiplicativity condition

$$x(mn) \ll |x(m)x(n)|$$
$$y(mn) \ll |y(m)y(n)|$$

holds for all natural numbers m and n.

Definitions and Assumptions

Our main result has two parts by using the following assumptions.

Divisor Bound Assumption: Assume that there exist $k_1, k_2, \ell_1, \ell_2 \geq 1$ such that

$$x(n) \ll \tau_{k_1}(n)(\log n)^{\ell_1},$$

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where $\tau_k(\cdot)$ is the k-fold divisor function given by the coefficients of $\zeta(s)^k$.

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GRH(Θ) Conjecture: There exists $\Theta \in [\frac{1}{2}, 1)$ such that for all $q \geqslant 1$ and for all Dirichlet characters χ modulo q, the Dirichlet L-functions $L(s, \chi)$ have no zeros in the region $\sigma > \Theta$.

Definitions and Assumptions

Let $\alpha \in \mathbb{C}$ such that $|\alpha| \leqslant \frac{1}{15 \log T}$ and define

$$s_\alpha(n) := n^\alpha.$$

For a natural number k, define

$$\Phi(s, k) := \prod_{p|k} \left(1 - p^{-s}\right), \quad (s \in \mathbb{C}).$$

Main Result

For $k, h \in \mathbb{N}$, define

$$\mathcal{F}_{\alpha,h,k}(T) := \frac{T}{2\pi}\left(\frac{\mathbb{1}_{k=1}}{h^{\alpha}}\frac{\zeta'}{\zeta}(1+\alpha) - \frac{\Lambda(k)}{h^{\alpha}\Phi(1+\alpha,k)} - \frac{k}{\varphi(k)}\Phi(\alpha,k)\zeta(1-\alpha)\frac{\left(\frac{T}{2\pi k}\right)^{-\alpha}}{1-\alpha}\right).$$

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Theorem (Kübra Benli, E. , Nathan Ng)

With the definitions and the assumptions above, we have

$$\begin{aligned} S(\alpha, T, X, Y) &= \sum_{0<\gamma\leq T}\zeta(\rho+\alpha)X(\rho)Y(1-\rho) \\ &= \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right)\sum_{n\leqslant N}\frac{(s_{-\alpha}*x)(n)y(n)}{n} - \frac{T}{2\pi}\sum_{n\leqslant N}\frac{(\Lambda * s_{-\alpha}*x)(n)y(n)}{n} \\ &\quad + \sum_{g\leqslant N}\sum_{\substack{h,k\leqslant N/g \\ (h,k)=1}}\frac{y(gh)x(gk)}{gkh}\mathcal{F}_{\alpha,h,k}(T) + \tilde{\mathcal{E}} \end{aligned}$$

where the error term $\tilde{\mathcal{E}}$ satisfies the following bounds.

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Theorem (Kübra Benli, E. , Nathan Ng)

1. *Under the Divisor Bound Assumption, we have*

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2. *Under the GRH*(Θ) *Conjecture, we have*

$$\begin{aligned}\tilde{\mathcal{E}} \ll\ & T^{\Theta+\varepsilon}\left(\left\|\frac{y(n)}{n^{\Theta}}\right\|_1 \left\|n^{1/2}x(n)(1*|y|)(n)\right\|_1\right)\\ &+ T^{\Theta+\varepsilon}\left(\left\|\frac{x(n)y(n)}{n}\right\|_1 \left\|\frac{y(n)}{n^{\Theta}}\right\|_1 \left\|\frac{x(n)}{n}\right\|_1 \left\|\frac{x(n)}{n^{2-\Theta}}\right\|_1\right)\\ &+ T^{\frac{1}{2}+\varepsilon}\left(\|x\|_1 \left\|\frac{y(n)}{n}\right\|_1 + \|y\|_1 \left\|\frac{x(n)}{n}\right\|_1\right)\end{aligned}$$

for any $\epsilon > 0$*.*

Main Result with Higher Derivatives

For $m \geqslant 1$, define

$$S_m(T,X,Y) := \frac{d}{d\alpha^m}(S(\alpha,T,X,Y))\bigg|_{\alpha=0} = \sum_{0<\gamma\leqslant T} \zeta^{(m)}(\rho)X(\rho)Y(1-\rho).$$

Since the error term in our main result for $S(\alpha,T,X,Y)$ is independent of α, we can apply the Cauchy Integral Formula to $S(\alpha,T,X,Y)$ to estimate the m^{th} derivative $S_m(T,X,Y)$.

Main Result with Higher Derivatives

Theorem (Kübra Benli, E. , Nathan Ng)

For $m \geqslant 1$, we have

$$S_m(T,X,Y) = \frac{(-1)^{m+1}}{m+1}\frac{T}{2\pi}\sum_{g\leqslant N}\sum_{h\leqslant N/g}\frac{y(gh)x(g)}{gh}\left(\mathcal{P}_{m+1}\left(\log\left(\frac{T}{2\pi}\right)\right)-\mathcal{Q}_{m+1}(\log h)\right)$$
$$+\frac{T}{2\pi}\sum_{g\leqslant N}\sum_{\substack{h,k\leqslant N/g\\ k\geqslant 2\\ (h,k)=1}}\frac{y(gh)x(gk)}{gkh}\left((-1)^{m+1}\mathcal{A}_m(h,k)+\mathcal{B}_m(k,T)\right)$$
$$+(-1)^m\frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right)\sum_{n\leqslant N}\frac{(\log^m * x)(n)y(n)}{n}$$
$$+(-1)^{m+1}\frac{T}{2\pi}\sum_{n\leqslant N}\frac{(\Lambda * \log^m * x)(n)y(n)}{n}+\tilde{\mathcal{E}}$$

where $\mathcal{P}_{m+1}$ and $\mathcal{Q}_{m+1}$ are monic polynomials of degree $m+1$ and $\mathcal{A}_m(h,k)$ and $\mathcal{B}_m(k,T)$ are some arithmetic weights.

Part III: Ideas in the Proofs

Sketch Proof

Our aim is to estimate the sum

$$S(\alpha, T, X, Y) = \sum_{0<\gamma\leq T} \zeta(\rho+\alpha)X(\rho)Y(1-\rho)$$

where T is large and $|\alpha| \leqslant \frac{1}{15 \log T}$.

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By the Residue Theorem, we have

$$S(\alpha, T, X, Y) = -\frac{1}{2\pi i} \int_{\mathscr{C}} \frac{\zeta'}{\zeta}(1-s)\zeta(s+\alpha)X(s)Y(1-s)\, ds.$$

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Let $\mathcal{S}_R$ and $\mathcal{S}_L$ denote the integrals over the right vertical line and the left vertical line of the contour $\mathscr{C}$, respectively.

Sketch Proof

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$$S(\alpha, T, X, Y) = \sum_{0<\gamma\leq T} \zeta(\rho+\alpha)X(\rho)Y(1-\rho)$$

where T is large and $|\alpha| \leqslant \frac{1}{15 \log T}$.

Let $\kappa := 1 + \frac{1}{\log T}$ and $\mathscr{C}$ be the positively oriented rectangle with vertices at $\kappa + i, \kappa + iT, 1 - \kappa + iT$ and $1 - \kappa + i$.

By the Residue Theorem, we have

$$S(\alpha, T, X, Y) = -\frac{1}{2\pi i} \int_{\mathscr{C}} \frac{\zeta'}{\zeta}(1-s)\zeta(s+\alpha)X(s)Y(1-s)\, ds.$$

Let $\mathcal{S}_R$ and $\mathcal{S}_L$ denote the integrals over the right vertical line and the left vertical line of the contour $\mathscr{C}$, respectively. Then, by controlling the contributions of the horizontal parts via the convexity bounds, we have

$$S(\alpha, T, X, Y) = \mathcal{S}_R + \mathcal{S}_L + +O\left(\mathfrak{e}_1\right)$$

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$$S(\alpha, T, X, Y) = \mathcal{S}_R + \mathcal{S}_L + +O\left(\mathfrak{e}_1\right)$$

where

$$\mathfrak{e}_1 \ll T^{\frac{1}{2}+\varepsilon}\left(\|x\|_1 \left\|\frac{y(n)}{n}\right\|_1 + \|y\|_1 \left\|\frac{x(n)}{n}\right\|_1\right).$$

Sketch Proof

For the contribution of the right edge

$$\mathcal{S}_R = -\frac{1}{2\pi i}\int_{\kappa+i}^{\kappa+iT}\frac{\zeta'}{\zeta}(1-s)\zeta(s+\alpha)X(s)Y(1-s)\,ds,$$

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we use

$$\frac{\zeta'}{\zeta}(1-s) = \frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(s) = -\log\left(\frac{|t|}{2\pi}\right) - \frac{\zeta'}{\zeta}(s) + O\left(|t|^{-1}\right)$$

where $\chi(\cdot)$ is the functional equation factor for the Riemann zeta function.

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where $\chi(\cdot)$ is the functional equation factor for the Riemann zeta function.

Then by using the underlying Dirichlet series of the integrand in $\mathcal{S}_R$, we have

$$\mathcal{S}_R = \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right)\sum_{n\leqslant N}\frac{\left(s_{-\alpha} * x\right)(n)y(n)}{n} - \frac{T}{2\pi}\sum_{n\leqslant N}\frac{\left(\Lambda * s_{-\alpha} * x\right)(n)y(n)}{n} + O\left(\mathfrak{e}_1\right).$$

Sketch Proof

For the contribution of the left edge, $\mathcal{S}_L$, we use the functional equation $\zeta(s+\alpha) = \chi(s+\alpha)\zeta(1-s-\alpha)$ to rewrite $\mathcal{S}_L$ as

$$\mathcal{S}_L = -\frac{1}{2\pi i}\int_{1-\kappa+iT}^{1-\kappa+i} \frac{\zeta'}{\zeta}(1-s)\chi(s+\alpha)\zeta(1-s-\alpha)X(s)Y(1-s)\,ds$$

$$= \overline{\frac{1}{2\pi}\int_1^T \frac{\zeta'}{\zeta}(\kappa+it)\chi(1-\kappa-it+\overline{\alpha})\zeta(\kappa+it-\overline{\alpha})\overline{X}(1-\kappa-it)\overline{Y}(\kappa+it)\,dt}$$

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where we use the notation $\overline{X}(s)=\sum_{n\leqslant N}\overline{x(n)}/n^s$ and $\overline{Y}(s)=\sum_{n\leqslant N}\overline{y(n)}/n^s$.

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$$\mathcal{S}_L(\gamma) := \frac{1}{2\pi}\int_1^T \frac{\zeta(\kappa+\gamma+it)}{\zeta(\kappa+it)}\chi(1-\kappa-it+\overline{\alpha})\zeta(\kappa+it-\overline{\alpha})\overline{X}(1-\kappa-it)\overline{Y}(\kappa+it)\,dt$$

for $\gamma \in \mathbb{C}$ with $|\gamma| \leqslant \frac{1}{15\log T}$.

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for $\gamma \in \mathbb{C}$ with $|\gamma| \leqslant \frac{1}{15\log T}$. Then we have

$$\mathcal{S}_L = \overline{\frac{d}{d\gamma}\mathcal{S}_L(\gamma)\Big|_{\gamma=0}}.$$

Sketch Proof

Now our aim is to estimate the integral

$$\mathcal{S}_L(\gamma) = \frac{1}{2\pi} \int_1^T \frac{\zeta(\kappa+\gamma+it)}{\zeta(\kappa+it)} \chi(1-\kappa-it+\overline{\alpha})\zeta(\kappa+it-\overline{\alpha})\overline{X}(1-\kappa-it)\overline{Y}(\kappa+it)\, dt.$$

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For $\Re(w) = \kappa - \Re(\overline{\alpha}) > 1$, define

$$A(w) := \frac{\zeta(w+\overline{\alpha}+\gamma)}{\zeta(w+\overline{\alpha})}\zeta(w)\overline{Y}(w+\overline{\alpha}) = \sum_{m=1}^{\infty}\frac{\sum_{\substack{m_1m_2m_3m_4=m\\ m_4\leqslant N}}\mu(m_1)m_1^{-\overline{\alpha}}m_2^{-\overline{\alpha}-\gamma}\overline{y(m_4)}m_4^{-\overline{\alpha}}}{m^w}$$

$$= \sum_{m=1}^{\infty}\frac{a(m)}{m^w}$$

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and

$$B(1-w) := \overline{X}(1-w-\overline{\alpha}) = \sum_{k\leqslant N}\frac{\overline{x(k)}k^{\overline{\alpha}}}{k^{1-w}} = \sum_{k\leqslant N}\frac{b(k)}{k^{1-w}}.$$

Then we have

$$\mathcal{S}_L(\gamma) = \frac{1}{2\pi i}\int_{\kappa+i-\overline{\alpha}}^{\kappa+iT-\overline{\alpha}}\chi(1-w)B(1-w)A(w)\,dw.$$

Sketch Proof

By using the stationary phase method, we have

$$\mathcal{S}_L(\gamma) = \sum_{k \leqslant N} \frac{b(k)}{k} \sum_{m \leqslant kT/2\pi} a(m) e\left(-m/k\right) + O\left(\mathfrak{e}_1\right)$$

where $e\left(-m/k\right) = e^{-2\pi i m/k}$.

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where $e\left(-m/k\right) = e^{-2\pi i m/k}$. For the inner sum above, we use the identity

$$e\left(-\frac{m}{k}\right) = \frac{\mu(k/(k,m))}{\phi(k/(k,m))} + \sum_{\substack{q|k \\ q>1}} \sum_{\psi \pmod q} \tau(\overline{\psi}) \sum_{\substack{d|m \\ d|k}} \psi\left(\frac{m}{d}\right) \delta(q,k,d,\psi)$$

where

$$\delta(q,k,d,\psi) = \sum_{\substack{e|d \\ e|k/q}} \frac{\mu(d/e)}{\phi(k/e)} \overline{\psi}\Big(-\frac{k}{eq}\Big) \psi\left(\frac{d}{e}\right) \mu\left(\frac{k}{eq}\right).$$

Sketch Proof

Let

$$\mathcal{M}(\gamma) := \sum_{k \leqslant N} \frac{b(k)}{k} \sum_{m \leqslant kT/2\pi} a(m) \frac{\mu(k/(m,k))}{\phi(k/(m,k))}$$

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Thus we have

$$\mathcal{S}_L(\gamma) = \mathcal{M}(\gamma) + \mathcal{E}(\gamma) + O(\mathfrak{e}_1)$$

and by differentiating with respect to γ via the Cauchy Integral Formula, we have

$$\mathcal{S}_L = \mathcal{M} + \mathcal{E} + O\left(\mathfrak{e}_1\right).$$

Now we state our results for the terms $\mathcal{M}$ and $\mathcal{E}$.

The term $\mathcal{M}$

Proposition

We have

$$\mathcal{M} = \sum_{g \leqslant N} \sum_{\substack{h,k \leqslant N/g \\ (h,k)=1}} \frac{y(gh)x(gk)}{ghk} \mathcal{F}_{\alpha,h,k}(T) + O\left(\mathfrak{e}_2\right)$$

where

$$\mathcal{F}_{\alpha,h,k}(T) = \frac{T}{2\pi}\left(\frac{\mathbb{1}_{k=1}}{h^\alpha}\frac{\zeta'}{\zeta}(1+\alpha) - \frac{\Lambda(k)}{h^\alpha \Phi(1+\alpha,k)} - \frac{k}{\varphi(k)}\Phi(\alpha,k)\zeta(1-\alpha)\frac{\left(\frac{T}{2\pi k}\right)^{-\alpha}}{1-\alpha}\right)$$

and

$$\mathfrak{e}_2 := \begin{cases} T\exp\left(-c\sqrt{\log T}\right) & \textit{on the Divisor Bound Assumption,} \\ T^{\Theta+\varepsilon}\left\|\frac{x(n)y(n)}{n}\right\|_1 \left\|\frac{y(n)}{n^\Theta}\right\|_1 \left\|\frac{x(n)}{n}\right\|_1 \left\|\frac{x(n)}{n^{2-\Theta}}\right\|_1 & \textit{on the GRH}(\Theta)\textit{ Conjecture} \end{cases}$$

for some positive constant c.

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for some positive constant c.

The proof of this result uses a decomposition lemma by Conrey, Ghosh and Gonek to find the generating series for the coefficients appearing in $\mathcal{M}(\gamma)$ and then we estimate the corresponding summatory function by Perron's formula.

The term $\mathcal{E}$

Proposition

We have

$$\mathcal{E} \ll \begin{cases} T(\log T)^{-A} & \text{on the Divisor Bound Assumption,} \\ T^{\Theta+\varepsilon} \left\| \frac{y(n)}{n^{\Theta}} \right\|_1 \left\| n^{1/2} x(n)(1 * |y|)(n) \right\|_1 & \text{on the GRH}(\Theta) \text{ Conjecture.} \end{cases}$$

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The proof of this result uses again the underlying generating series and Perron's formula in the conjectural case. But in the case where we assume the Divisor Bound Assumption only, we use Heath-Brown's combinatorial decomposition of $\mu(n)$ and the Large Sieve Inequality as utilized in the work of Heap, Li and Zhao.

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By combining the estimates above, we obtain our main result on the discrete mean value

$$S(\alpha, T, X, Y) = \sum_{0<\gamma\leq T} \zeta(\rho+\alpha) X(\rho) Y(1-\rho).$$

Recall: The Main Result

For $k, h \in \mathbb{N}$, recall that

$$\mathcal{F}_{\alpha,h,k}(T) = \frac{T}{2\pi}\left(\frac{\mathbb{1}_{k=1}}{h^\alpha}\frac{\zeta'}{\zeta}(1+\alpha) - \frac{\Lambda(k)}{h^\alpha \Phi(1+\alpha,k)} - \frac{k}{\varphi(k)}\Phi(\alpha,k)\zeta(1-\alpha)\frac{\left(\frac{T}{2\pi k}\right)^{-\alpha}}{1-\alpha}\right).$$

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Theorem (Kübra Benli, E. , Nathan Ng)

With the definitions and the assumptions above, we have

$$\begin{aligned} S(\alpha,T,X,Y) &= \sum_{0<\gamma\leq T}\zeta(\rho+\alpha)X(\rho)Y(1-\rho) \\ &= \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right)\sum_{n\leqslant N}\frac{(s_{-\alpha}*x)(n)y(n)}{n} - \frac{T}{2\pi}\sum_{n\leqslant N}\frac{(\Lambda*s_{-\alpha}*x)(n)y(n)}{n} \\ &\quad + \sum_{g\leqslant N}\sum_{\substack{h,k\leqslant N/g \\ (h,k)=1}}\frac{y(gh)x(gk)}{gkh}\mathcal{F}_{\alpha,h,k}(T) + \tilde{\mathcal{E}}. \end{aligned}$$

Recall: Main Result with Higher Derivatives

By using the Cauchy Integral Formula and the previous result, we obtain the following estimate for higher derivatives.

Theorem (Kübra Benli, E. , Nathan Ng)

For $m \geqslant 1$, we have

$$\begin{aligned}
S_m(T,X,Y) &= \frac{(-1)^{m+1}}{m+1}\frac{T}{2\pi}\sum_{g\leqslant N}\sum_{h\leqslant N/g}\frac{y(gh)x(g)}{gh}\left(\mathcal{P}_{m+1}\left(\log\left(\frac{T}{2\pi}\right)\right)-\mathcal{Q}_{m+1}(\log h)\right)\\
&+\frac{T}{2\pi}\sum_{g\leqslant N}\sum_{\substack{h,k\leqslant N/g\\ k\geqslant 2\\ (h,k)=1}}\frac{y(gh)x(gk)}{gkh}\left((-1)^{m+1}\mathcal{A}_m(h,k)+\mathcal{B}_m(k,T)\right)\\
&+(-1)^m\frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right)\sum_{n\leqslant N}\frac{(\log^m *x)(n)y(n)}{n}\\
&+(-1)^{m+1}\frac{T}{2\pi}\sum_{n\leqslant N}\frac{(\Lambda*\log^m *x)(n)y(n)}{n}+\tilde{\mathcal{E}}
\end{aligned}$$

where $\mathcal{P}_{m+1}$ and $\mathcal{Q}_{m+1}$ are monic polynomials of degree $m+1$ and $\mathcal{A}_m(h,k)$ and $\mathcal{B}_m(k,T)$ are some arithmetic weights.

Sketch Proof of the Corollary

Now our aim to obtain the lower bound

$$\sum_{0<\gamma\leqslant T} |\zeta^{(m)}(\rho)|^{2k} \gg T(\log T)^{k^2+2km+1}.$$

for $k, m \in \mathbb{N}$ under the assumption of the Riemann Hypothesis.

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for $k, m \in \mathbb{N}$ under the assumption of the Riemann Hypothesis. For $0 < \vartheta < \frac{1}{2}$ and sufficiently large T, let $N = \xi^k = T^\vartheta$. Define

$$\mathcal{C}_\xi(s) := \sum_{n\leqslant\xi} \frac{1}{n^s}.$$

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$$\mathcal{C}_\xi(s) := \sum_{n\leqslant\xi} \frac{1}{n^s}.$$

Observe that

$$\Sigma_1 := \sum_{0<\gamma\leqslant T} \zeta^{(m)}(\rho)\mathcal{C}_\xi(\rho)^{k-1}\overline{\mathcal{C}_\xi(\rho)}^k = \sum_{0<\gamma\leqslant T} \zeta^{(m)}(\rho)\mathcal{C}_\xi(\rho)^{k-1}\mathcal{C}_\xi(1-\rho)^k$$

by the assumption of the Riemann Hypothesis.

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by the assumption of the Riemann Hypothesis. By Hölder's inequality, we have

$$|\Sigma_1| \leqslant \left(\sum_{0<\gamma\leqslant T} \left|\zeta^{(m)}(\rho)\right|^{2k}\right)^{\frac{1}{2k}} \left(\sum_{0<\gamma\leqslant T} \left(|\mathcal{C}_\xi(\rho)|^{2k-1}\right)^{\frac{2k}{2k-1}}\right)^{\frac{2k-1}{2k}}.$$

Sketch Proof of the Corollary

Thus, by taking the $2k^{\text{th}}$ power of both sides, we have

$$\left|\Sigma_1\right|^{2k} \leqslant \left(\sum_{0<\gamma<T} \left|\zeta^{(m)}(\rho)\right|^{2k} \right) \Sigma_2^{2k-1}$$

where

$$\Sigma_2 := \sum_{0<\gamma\leqslant T} \left|\mathcal{C}_\xi(\rho)\right|^{2k}.$$

Thus, by taking the $2k^{\text{th}}$ power of both sides, we have

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where

$$\Sigma_2 := \sum_{0<\gamma\leqslant T} |\mathcal{C}_\xi(\rho)|^{2k} .$$

This gives the lower bound

$$\sum_{0<\gamma\leqslant T} \left| \zeta^{(m)}(\rho) \right|^{2k} \geqslant \frac{|\Sigma_1|^{2k}}{\Sigma_2^{2k-1}}.$$

Sketch Proof of the Corollary

By using our main result concerning higher derivatives, we have

$$\Sigma_1 \gg T\left(\log T\right)^{k^2+m+1}.$$

By a result of Milinovich and Ng, we have

$$\Sigma_2 \ll T\left(\log T\right)^{k^2+1}.$$

Hence

$$\sum_{0<\gamma\leqslant T}\left|\zeta^{(m)}(\rho)\right|^{2k} \geqslant \frac{|\Sigma_1|^{2k}}{\Sigma_2^{2k-1}} \gg \frac{T^{2k}\left(\log T\right)^{2k\left(k^2+m+1\right)}}{T^{2k-1}\left(\log T\right)^{(2k-1)\left(k^2+1\right)}} = T(\log T)^{k^2+2km+1}.$$

THANK YOU!

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