Explicit bounds for Möbius sums and $1/|\zeta(s)|$

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Mertens function

Definition

The Möbius function is defined for all positive integers n as

$$\mu(n) = \begin{cases} +1 & \text{if} \quad n \text{ square-free and has } \textit{even} \text{ number of prime factors} \\ -1 & \text{if} \quad n \text{ square-free and has } \textit{odd} \text{ number of prime factors} \\ 0 & \text{if} \quad n \text{ contains a square.} \end{cases}$$

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$$M(x) = \sum_{n \leq x} \mu(n),$$

where $\mu(n)$ is the Möbius function.

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To better understand $M(x)$, we seek explicit bounds on it.

Some current results

- Helfgott and Thompson (2023) verified computationally:

$$|M(x)| < 0.571\sqrt{x} \qquad 33 \leq x \leq 10^{23}$$

- Cohen, Dress, El Marraki (2007) proved:

$$|M(x)| < \frac{x}{4345} \qquad x \geq 2160535. \tag{1}$$

- Ramaré (2013) proved:

$$|M(x)| < \frac{0.013x}{\log x} - \frac{0.118x}{(\log x)^2} \qquad x \geq 1078853.$$

- El Marraki (1995) proved explicit bounds of form:

$$|M(x)| < \frac{C_0 x}{(\log x)^k}, \qquad (\text{any } k \geq 0). \tag{2}$$

- Chalker (2019) proved explicit bounds of form:

$$|M(x)| < C_1 x \log x \exp\left(-C_2\sqrt{\log x}\right), \tag{3}$$

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Our results

We prove, the first explicit version of Walfisz (1963):

$$M(x) = O\left(x \exp(-C(\log x)^{3/5}(\log \log x)^{-1/5})\right),$$

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Our method uses a Perron formula argument and bounds for $1/\zeta(s)$:

$$M(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s\zeta(s)}ds \qquad (c > 1).$$

Bounds for $1/\zeta(s)$

Classically, there are three types of zero-free regions. Let c_1, c_2, c_3 be constants. We have that $\zeta(\sigma + it) \neq 0$ in the region

$$\sigma \geq 1 - \frac{1}{c_1 \log t} \quad \text{(de la Vallée Poussin)},$$

$$\sigma \geq 1 - \frac{\log\log t}{c_2 \log t} \quad \text{(Littlewood)},$$

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We make the improvements:

$$\frac{1}{\zeta(s)} \ll (\log t)^{11/12} \qquad \text{and} \qquad \frac{1}{\zeta(s)} \ll (\log t)^{2/3} (\log \log t)^{1/4}.$$

Sketch proof for the case $(\log t)$

First obtain a uniform bound when $1-\delta \leq \sigma \leq 1+\delta_1$:

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Apply to the identity

$$\log\left|\frac{1}{\zeta(\sigma+it)}\right| = -\Re \log \zeta(1+\delta_1+it) + \int_{\sigma}^{1+\delta_1} \Re \frac{\zeta'}{\zeta}(x+it)dx.$$

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For $1-\delta \le \sigma \le 1+\delta_1$, we arrive at

$$\left|\frac{1}{\zeta(\sigma+it)}\right| \le \left|\frac{1}{\zeta(1+\delta_1+it)}\right| \exp\left((\delta_1+\delta)C\log t\right)$$
$$= \left|\frac{1}{\zeta(1+\delta_1+it)}\right| \exp(C_1), \quad (5)$$

by choosing $\delta, \delta_1 = O(1/\log t)$.

The order of the left-hand side of (5) is equivalent to the order of

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Estimating $1/\zeta(s)$ when $\sigma > 1$

Typically, a trivial bound for $\sigma > 1$ would give

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$$\text{i.e.,} \qquad \left|\frac{1}{\zeta(1+\delta_1+it)}\right| \leq |\zeta(1+\delta_1)|^{3/4}|\zeta(1+\delta_1+2it)|^{1/4}$$
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by (6). Note that applying (6) again would give an overall $O(\log t)$ bound. Our goal is to improve this.

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So far, we have for $1 - \delta \leq \sigma \leq 1 + \delta_1 \leq 2$,

$$\left|\frac{1}{\zeta(\sigma + it)}\right| \leq C_3 \left|\frac{1}{\zeta(1 + \delta_1 + it)}\right| \ll (\log t)^{3/4} |\zeta(1 + \delta_1 + 2it)|^{1/4}. \quad (7)$$

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Instead of using trivial bounds, we use the Phragmén–Lindelöf principle to combine the bounds

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so that $\zeta(s) \ll (\log t)^{2/3}$ for $1 \le \sigma \le 2$.

Finally, apply to (7):

$$\left| \frac{1}{\zeta(s)} \right| \ll (\log t)^{3/4} ((\log t)^{2/3})^{1/4} = (\log t)^{11/12}.$$