The Shanks–Rényi prime number race problem

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Based on joint works with Kevin Ford, Adam Harper and Sergei Konyagin

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Chebyshev's observation

In 1853, Chebyshev wrote a letter to Fuss with the following statement

There is a notable difference in the splitting of the primes between the two forms $4n+3$, $4n+1$*: the first form contains a lot more than the second.*

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Table: The number of primes of the form $4n+3$ and $4n+1$ up to x (from A. Granville and G. Martin, "prime number races", Amer. Math. Monthly 113 (2006), no. 1, 1–33.)

x	$\pi(x;4,3)$	$\pi(x;4,1)$
100	13	11
500	50	44
1000	87	80
5000	339	329
10,000	619	609
50,000	2583	2549
100,000	4808	4783

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Theorem (Littlewood 1914)

The difference $\pi(x;4,1) - \pi(x;4,3)$ changes sign for infinitely many integers x.

The race modulo 3

Table: The number of primes of the form $3n+2$ and $3n+1$ up to x

x	$\pi(x;3,2)$	$\pi(x;3,1)$
100	13	11
1000	87	80
10,000	617	611
100,000	4807	4784
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Bays and Hudson (Christmas Day of 1976): $\pi(x;3,1) > \pi(x;3,2)$ for the first time when $x = 608{,}981{,}813{,}029 \approx 6 \times 10^{11}$.

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1. Will each of the players take the lead for infinitely many integers x?
2. Will all $r!$ orderings of the players occur for infinitely many integers x?
3. What is the "probability" that a particular ordering $\pi(x; q, a_1) > \pi(x; q, a_2) > \cdots > \pi(x; q, a_r)$ occurs?

The two-way race

Haselgrove's condition for the modulus q

For all characters χ modulo q we have $L(s, \chi) \neq 0$ for all $s \in (0, 1)$.

Theorem (Knapowski and Turán, 1962)

Assume Haselgrove's condition for the modulus q. For any $(a, q) = 1$ such that $a \not\equiv 1 \pmod{q}$, the quantity $\pi(x; q, a) - \pi(x; q, 1)$ changes sign for infinitely many integers x.

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- Rumely (1993) : Haselgrove's condition is true for all $q \leq 72$.
- Sneed (2009) : Haselgrove's condition is true for all $q \leq 100$.

Theorem (Kátai, 1964)

Assume Haselgrove's condition for the modulus q. If $a \not\equiv b \pmod q$ are both squares, or both non-squares modulo q, then $\pi(x; q, a) - \pi(x; q, b)$ changes sign for infinitely many integers x.

Theorem (Kátai, 1964)

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Theorem (Sneed, 2009)

For all $q \leq 100$ and all $(ab, q) = 1$, $\pi(x; q, a) - \pi(x; q, b)$ changes sign for infinitely many integers x.

Races with 3 or more competitors

Theorem (Kaczorowski, 1993)

Assume GRH. Let $q \geq 3$. There exist infinitely many integers x such that

$$\pi(x; q, 1) > \max_{a \not\equiv 1 \bmod q} \pi(x; q, a).$$

The same is true for $\pi(x; q, 1) < \min_{a \not\equiv 1 \bmod q} \pi(x; q, a)$.

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Theorem (Kaczorowski, 1996)

- Assume GRH. For each $q \geq 5$, $q \neq 6$ one can construct an explicit $\varphi(q)$-race such that $\pi(x; q, 1) > \pi(x; q, b_2) \cdots > \pi(x; q, b_{\varphi(q)})$ holds for infinitely many integers x.

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- For example $\pi(x; 7, 1) > \pi(x; 7, 5) > \pi(x; 7, 6) > \pi(x; 7, 3) > \pi(x; 7, 4) > \pi(x; 7, 2)$ holds for infinitely many integers x.

Barriers : The work of Ford and Konyagin

Theorem (Ford and Konyagin, 2002)

Let $q \geq 5$ and a_1, a_2, a_3 be distinct residue classes mod q that are coprime to q. Let τ be arbitrarily large.

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Theorem (Ford and Konyagin, 2003)

Fix $q \geq 5$ and an arbitrarily large τ. Let $4 \leq r \leq \varphi(q)$ and $a_1, \ldots, a_r$ be distinct residue classes mod q that are coprime to q. There exists a **finite set** $\mathcal{B} = \{\sigma + it : 1/2 < \sigma \leq 1 \text{ and } t \geq \tau\}$, such that if certain L-functions $L(s, \chi)$ with $\chi \bmod q$ have zeros (with certain multiplicities) in $\mathcal{B}$ then at most $r(r-1)$ of the $r!$ orderings of the functions $\pi(x; q, a_1), \pi(x; q, a_2), \ldots, \pi(x; q, a_r)$ occur for large x.

Densities in the prime number race problem

Conjecture (Knapowski-Turàn 1962)

As $X \to \infty$, the percentage of integers $x \leq X$ for which $\pi(x;4,1) > \pi(x;4,3)$ goes to to 0%.

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The following table gives, for X in various ranges, the maximum percentage of values of $x \leq X$ for which $\pi(x; 4, 1) > \pi(x; 4, 3)$:

Table:

For X in the range	Maximum percentage of such $x \leq X$
$0 - 10^7$	2.6%
$10^7 - 10^8$	0.6%
$10^7 - 10^8$	0.6%
$10^8 - 10^9$	0.1%
$10^9 - 10^{10}$	1.6%
$10^{10} - 10^{11}$	2.8%

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Theorem (Ford, Konyagin and L. 2013)

We can construct a set $\mathcal{B} = \{\sigma + it : 0 \leq \sigma \leq 1 \text{ and } \sigma \neq 1/2\}$ such that if $L(s, \chi_1)$ has zeros (with certain multiplicities) in $\mathcal{B}$ (where χ_1 is the non-principal character modulo 4), then the Knapowski-Turàn Conjecture is true!

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- They can be arbitrarily close to the critical line $\text{Re}(s) = 1/2$.

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- The hypothetical zeros of $L(s, \chi_1)$ can be chosen with arbitrarily large imaginary parts.
- They can be arbitrarily close to the critical line $\text{Re}(s) = 1/2$.
- $\mathcal{B}$ is a very "thin" set. Indeed, our construction involves $O\left((\log T)^{5/4}\right)$ zeros (counted with multiplicity) with imaginary part less than T.

Theorem (Kaczorowski 1993)

Assume GRH. For any $(a, q) = 1$ such that $a \not\equiv 1 \bmod q$, the set of positive integers x for which $\pi(x; q, 1) > \pi(x; q, a)$ has a positive lower density.

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Let a, b be distinct reduced residue classes modulo q. We can construct a set $\mathcal{B} = \{\sigma + it : 0 \le \sigma \le 1 \text{ and } \sigma \ne 1/2\}$ such that if a certain $L(s, \chi)$ has zeros (with certain multiplicities) in $\mathcal{B}$, then the percentage of integers $x \le X$ for which $\pi(x; q, a) > \pi(x; q, b)$ goes to 0%.

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There seem to be "more" primes of the form $qn + a$ than of the form $qn + b$ if a is **non-square** and b is a **square** modulo q.

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Table: The number of primes of the form $10n + j$ up to x

x	Last Digit 1	Last Digit 3	Last Digit 7	Last Digit 9
1000	40	42	46	38
10,000	306	310	308	303
100,000	2387	2402	2411	2390
1,000,000	19,617	19,665	19,621	19,593

Table: The number of primes of the form $8n+j$ up to x

x	$\pi(x;8,1)$	$\pi(x;8,3)$	$\pi(x;8,5)$	$\pi(x;8,7)$
1000	37	44	43	43
10,000	295	311	314	308
100,000	2384	2409	2399	2399
1,000,000	19,552	19,653	19,623	19,669

Measuring the bias: the work of Rubinstein-Sarnak

Theorem (Kaczorowski 1993)

Assume the Generalized Riemann Hypothesis. The quantity

$$\frac{1}{X}|\{x \le X : \pi(x;4,3) > \pi(x;4,1)\}|$$

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- The natural density is not the correct way to measure the bias!
- The adequate measure to use is the **logarithmic measure**.
- Need to use the Generalized Riemann Hypothesis and the *Linear Independence Conjecture LI*, which is the assumption that the positive imaginary parts of the zeros of the associated Dirichlet L-functions are linearly independent over the rational numbers.

Theorem (Rubinstein and Sarnak, 1994)

Assume GRH and LI. As $X \to \infty$,

$$\frac{1}{\log X} \sum_{\substack{x \le X \\ \pi(x;4,3) > \pi(x;4,1)}} \frac{1}{x} \to 0.9959\ldots$$

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Theorem (Rubinstein and Sarnak, 1994)

Assume GRH and LI. Let a, b be distinct reduced residue classes modulo q. The *logarithmic density*

$$\delta(q; a, b) := \lim_{X \to \infty} \frac{1}{\log X} \int_{\substack{t \in [2,X] \\ \pi(t;q,a) > \pi(t;q,b)}} \frac{dt}{t},$$

exists and is positive.

$\delta(q; a, b)$ is **"the probability"** that $\pi(x; q, a) > \pi(x; q, b)$.

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- $\delta(4; 3, 1) = 0.9959\ldots$ and $\delta(3; 2, 1) = 0.9990\ldots$.

- Let $2 \leq r \leq \varphi(q)$ and $a_1, \ldots, a_r$ be distinct residue classes modulo q which are coprime to q.
- Let $P(q; a_1, \ldots, a_r)$ be the set of positive integer x such that

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- The race $\{q; a_1, \dots, a_r\}$ is said to be **unbiased** if for every permutation σ of the set $\{1, 2, \dots, r\}$ we have

$$\delta(q; a_{\sigma(1)}, \dots, a_{\sigma(r)}) = \delta(q; a_1, \dots, a_r) = \frac{1}{r!}.$$

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If a race $\{q; a_1, \ldots, a_r\}$ is unbiased, then all the a_i must be either squares or either non-squares modulo q.

- **Guess**: If $a_1, \ldots, a_r$ are all squares or all non-squares modulo q, then the race $\{q; a_1, \ldots, a_r\}$ is unbiased.

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Assume GRH and LI. Fix $r \geq 3$. There exists a constant $q_0(r)$ such that if $q \geq q_0(r)$ then

- There exist distinct residue classes $a_1, \ldots, a_r$ mod q, with $(a_i, q) = 1$, $a_1, \ldots, a_r$ are **squares** modulo q and the race $\{q; a_1, \ldots, a_r\}$ is biased.

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The size of $\delta(q; a_1, \ldots, a_r)$ when r is fixed and q is large

$$\delta(4; 3, 1) = 0.9959\ldots \text{ (Rubinstein-Sarnak, 1994)}$$
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More generally if $r \geq 2$ is fixed, then

$$\max_{a_1, a_2, \ldots, a_r \bmod q} \left| \delta(q; a_1, \ldots, a_r) - \frac{1}{r!} \right| \to 0 \text{ as } q \to \infty.$$

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$$\Delta_r(q) := \max_{a_1,a_2,\ldots,a_r \bmod q} \left| \delta(q; a_1, \ldots, a_r) - \frac{1}{r!} \right|.$$

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$$\Delta_2(q) \sim \frac{\rho(q)}{2\sqrt{\pi\varphi(q)\log q}} = \frac{1}{q^{1/2+o(1)}},$$

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Theorem (L, 2013)

Assume GRH and LI. Let $r \geq 3$ be a fixed integer. There exist positive constants $c_1(r)$, $c_2(r)$ and a positive integer q_0 such that, if $q \geq q_0$ then

$$\frac{c_1(r)}{\log q} \leq \Delta_r(q) \leq \frac{c_2(r)}{\log q}.$$

The random verctor associated to a prime number race

- For a non-principal Dirichlet character $\chi \neq \chi_0$, let $\{1/2 + i\gamma_\chi\}$ be the sequence of non-trivial zeros of the Dirichlet L-function $L(s, \chi)$, and let $S_q = \cup_{\chi \neq \chi_0 \bmod q} \{\gamma_\chi : \gamma_\chi > 0\}$.

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- Let $\{U(\gamma_\chi)\}_{\gamma_\chi \in S_q}$ be a sequence of independent random variables uniformly distributed on the unit circle $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$.
- We consider the random variables

$$X(q,a) = -c_q(a) + \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 0} \frac{2\mathrm{Re}(\chi(a)U(\gamma_\chi))}{\sqrt{\frac{1}{4} + \gamma_\chi^2}},$$

where

$$c_q(a) := -1 + |\{b \bmod q : b^2 \equiv a \bmod q\}| = q^{o(1)}.$$

Theorem (Rubinstein and Sarnak, 1994)

Assume GRH and LI. Then we have

$$\delta(q; a_1, \dots, a_r) = \mathbb{P}\Big(X(q, a_1) > X(q, a_2) > \cdots > X(q, a_r)\Big).$$

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- **The mean vector** of $\Big(X(q, a_1), X(q, a_2), \ldots, > X(q, a_r)\Big)$ equals $(-c_q(a_1), \ldots, -c_q(a_r))$ and is responsible for Chebyshev's Bias (if $r = 2$).

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$$\mathrm{Cov}_{q;a_1,\ldots,a_r}(j, k) \begin{cases} \sim \varphi(q) \log q & \text{if } j = k \\ = O(\varphi(q)) & \text{if } j \neq k. \end{cases}$$

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- The random variables $X(q, a)$ are **weakly correlated**, and their correlations govern the behavior of $\Delta_r(q)$ for $r \geq 3$.

The size of $\delta(q; a_1, \ldots, a_r)$ when $r \to \infty$ as $q \to \infty$

Conjecture (Feuerverger-Martin, 2000)

There exist a function $r_0(q)$ with $r_0(q) \to \infty$ as $q \to \infty$, such that for any integer $r \leq r_0(q)$ we have

$$\lim_{q \to \infty} \max_{a_1, \ldots, a_r \bmod q} |r! \delta(q; a_1, \ldots, a_r) - 1| = 0,$$

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$$\delta(q; a_1, \ldots, a_r) = \frac{1}{r!} \left(1 + O\left(\frac{r^2}{\log q} \right) \right).$$

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1. If $2 \le r \le (\log q)^{1-\varepsilon}$, then uniformly for all r-tuples $(a_1, \ldots, a_r)$ of distinct reduced residue classes modulo q we have

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2. If $(\log q)^{1+\varepsilon} \le r \le \varphi(q)$, then there exist r-tuples $(a_1, \ldots, a_r), (b_1, \ldots, b_r)$ for which we have as $q \to \infty$
$$r! \cdot \delta(q; a_1, \ldots, a_r) \to 0$$
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The main ingredient of the proof is a harmonic analysis estimate related to the Hardy-Littlewood circle method, and inspired by work of Bourgain. This is used to control the average size of the **correlations** of the random variables $X(q, a_1), \ldots, X(q, a_r)$.

Theorem (Ford, Harper, and L. 2019)

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$$\delta(q; a_1, \ldots, a_r) \le \exp\left(-\frac{\min\{r, \varphi(q)^{1/50}\}}{C \log q}\right) \frac{1}{r!},$$

and

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- This establishes the second part of Ford and Lamzouri's Conjecture (under GRH and LI) as soon as $r/\log q \to \infty$.

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- Let $1 \leq k \leq r/2 \leq \varphi(q)/2$, and define $\delta_k(q; a_1, \dots, a_r)$ to be the logarithmic density of the set of $x \geq 2$ such that

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- Note that $\delta_k(q; a_1, \dots, a_r)$ is **the probability** that $X(q, a_1) > X(q, a_2) > \dots > X(q, a_k) > \max_{k+1 \leq j \leq r} X(q, a_j)$.

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- One can show that $\max_{k+1 \le j \le r} Z(q, a_j)$ is around $\sqrt{2 \log r}$ with high probability.

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- We finally choose k such that $k \log r / \log q \to \infty$ as $q \to \infty$.

Other directions

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- Chebyshev's bias in the number field setting (for Frobenius elements in Galois extensions): Ng (2000), Fiorilli-Jouve (2020), Bailleul (2021), Hayani (2024), ...

Thank you for your attention!