Unbalanced Optimal Transport: Convex Relaxation and Dynamic Perspectives

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Outline

Unbalanced Optimal Transport starting from Dirac masses

X_i Polish topological spaces (the topology is induced by a separable and complete metric).

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$$\boxed{h(x_0, r_0; x_1, r_1) := \mathrm{UOT}_{\mathrm{Dirac}}(r_0\delta_{x_0}, r_1\delta_{x_1}) \quad x_i \in X_i,\ r_i \geqslant 0}$$

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Simplifying assumptions: for every x_0, x_1

$$\begin{cases} h(x_0, r_0; x_1, 0) \quad \text{is independent of } x_1, \quad h(x_0, 0; x_1, r_1) \quad \text{is independent of } x_0. \\ (r_0, r_1) \mapsto h(x_0, r_0; x_1, r_1) \quad \text{is } \textbf{positively } 1\textbf{-homogeneous and convex} \end{cases}$$

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Cone space: identify all the points $(x, 0)$ **with the vertex** $\mathfrak{o}$ (they correspond to the null measure $0\delta_x = 0$)

$$\mathfrak{C}[X] := \big(X \times [0, \infty)\big)/\sim, \quad (x', r') \sim (x'', r'') \quad \Leftrightarrow \quad \begin{cases} x' = x'',\ r' = r'' \neq 0, \\ r' = r'' = 0 \end{cases}$$

Examples

The Balanced OT case: $c : X_0 \times X_1 \to \mathbb{R}$ is a cost function,

$$h(x_0, r_0; x_1, r_1) = \begin{cases} rc(x_0, x_1) & \text{if } r_0 = r_1 = r; \\ +\infty & \text{if } r_0 \neq r_1 \end{cases}$$

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The Hellinger-Kakutani case:

$$h(x_0, r_0; x_1, r_1) = \begin{cases} (\sqrt{r_0} - \sqrt{r_1})^2 = r_0 + r_1 - 2\sqrt{r_0 r_1} & \text{if } x_0 = x_1 \\ +\infty & \text{if } x_0 \neq x_1 \end{cases}$$

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The Entropic Unbalanced Cost

$$\begin{aligned} h(x_0, r_0; x_1, r_1) &= r_0 + r_1 - 2\sqrt{r_0 r_1} e^{-c(x_0, x_1)} \\ &= \left(\sqrt{r}_0 - \sqrt{r}_1\right)^2 + 2\sqrt{r_0 r_1}\left(1 - e^{-c(x_0, x_1)}\right). \end{aligned}$$

What is the most natural way (from the convex analysis viewpoint) to extend $\mathrm{UOT}_{\mathrm{Dirac}}$

$$\boxed{h(x_0, r_0; x_1, r_1) := \mathrm{UOT}_{\mathrm{Dirac}}(r_0\delta_{x_0}, r_1\delta_{x_1}) \quad x_i \in X_i,\ r_i \geqslant 0}$$

to a function in $\mathcal{M}(X_0) \times \mathcal{M}(X_1)$?

Unbalanced Optimal Transport as convex envelope

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Γ-relaxation of $\mathsf{UOT}_{\mathsf{Dirac}}$: the largest convex and l.s.c. functional $\Gamma\text{-}\mathsf{UOT}_{\mathsf{Dirac}} : \mathcal{M}(X_0) \times \mathcal{M}(X_1) \to [0, +\infty]$ dominated by $\mathsf{UOT}_{\mathsf{Dirac}}$:

$$\begin{cases} \Gamma\text{-}\mathsf{UOT}_{\mathsf{Dirac}}(r_0\delta_{x_0}, r_1\delta_{x_1}) \leqslant \mathsf{UOT}_{\mathsf{Dirac}}(r_0\delta_{x_0}, r_1\delta_{x_1}) \quad \text{for every } r_i \geqslant 0,\, x_i \in X_i \\ \hat{\mathsf{UOT}} \text{convex, l.s.c.,}\ \hat{\mathsf{UOT}} \leqslant \mathsf{UOT}_{\mathsf{Dirac}} \quad \Rightarrow \quad \hat{\mathsf{UOT}} \leqslant \Gamma\text{-}\mathsf{UOT}_{\mathsf{Dirac}}. \end{cases}$$

Two equivalent constructions

If $\mathscr{F} : V \to (-\infty, +\infty]$ is a given function, defined in a vector space V in duality with V', its Γ-regularization can be characterized in two equivalent ways:

- Using the **Legendre transform thanks to Fenchel-Moreau Theorem**:

$$\mathscr{F}^*(\phi) := \sup_{v \in V} \langle \phi, v \rangle - \mathscr{F}(v), \qquad \phi \in V'$$

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- Computing the **convex envelope**:

$$\mathrm{co}\mathscr{F}(v) := \inf \Big\{ \sum_i \alpha_i \mathscr{F}(v_i) : \alpha_i \geqslant 0, \ \sum_i \alpha_i = 1, \ \sum_i \alpha_i v_i = v \Big\}$$

  and then taking the **l.s.c. relaxation** of $\mathrm{co}\mathscr{F}(v)$. If $\mathscr{F}$ is coercive we have the integral description

$$\Gamma\text{-}\mathscr{F}(v) = \min \Big\{ \int_V \mathscr{F}(w) \, \mathrm{d}\alpha(w) : \alpha \in \mathcal{P}(V), \ \int_V w \, \mathrm{d}\alpha(w) = v \Big\}.$$

Convex duality

Γ-UOT$_{\text{Dirac}}$ can be computed by **Legendre transform thanks to Fenchel-Moreau Theorem**, using the duality between $\mathcal{M}(X)$ and $C_b(X)$.

$$\mathrm{UOT}^*_{\text{Dirac}}(\phi_0, \phi_1) = \sup \left\{ r_0\phi_0(x_0) + r_1\phi_1(x_1) - h(x_0, r_0; x_1, r_1) : r_i \geqslant 0,\ x_i \in X_i \right\}$$

$$= \begin{cases} 0 & \text{if } r_0\phi_0(x_0) + r_1\phi_1(x_1) \leqslant h(x_0, r_0; x_1, r_1), \\ +\infty & \text{otherwise} \end{cases}$$

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UOT$^*_{\text{Dirac}}$ is just the **indicator function of a convex set $K[h]$ of admissible Kantorovich potentials $(\phi_0, \phi_1) \in C_b(X_0) \times C_b(X_1)$.**

The dual Kantorovich formulation of Unbalanced Optimal Transport

$$\Gamma\text{-UOT}_{\text{Dirac}}(\mu_0, \mu_1) = \mathrm{UOT}^{**}_{\text{Dirac}}(\mu_0, \mu_1) =$$
$$= \sup \left\{ \int \phi_0 \, d\mu_0 + \int \phi_1 \, d\mu_1 : (\phi_0, \phi_1) \in K[h] \right\}.$$

Primal formulation

How to represent convex combinations of pair of Dirac masses?

Given $\alpha_k \geqslant 0$, $\sum_k \alpha_k = 1$, we may consider

$$(\mu_0, \mu_1) = \sum_k \alpha_k (r_{0,k}\delta_{x_{0,k}}, r_{1,k}\delta_{x_{1,k}})$$

$$\rightsquigarrow \Gamma\text{-UOT}_{\text{Dirac}}(\mu_0, \mu_1) \leqslant \sum_k \alpha_k h(x_{0,k}, r_{0,k}; x_{1,k}, r_{1,k}) = \int h \, d\boldsymbol{\alpha}$$

$$\rightsquigarrow \boldsymbol{\alpha} = \sum_k \alpha_k \delta_{(x_{0,k}, r_{0,k}; x_{1,k} r_{1,k})} \in \mathcal{P}(X_0 \times \mathbb{R}_+ \times X_1 \times \mathbb{R}_+)$$

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Constraints:

$$\mu_0(A) = \sum_k \alpha_k r_{0,k} \delta_{x_{0,k}}(A) = \int_{A \times \mathbb{R}_+ \times X_1 \times \mathbb{R}_+} r_0 \, d\boldsymbol{\alpha}(x_0, r_0; x_1, r_1)$$

$$= \mathfrak{h}_0 \boldsymbol{\alpha}(A)$$

$$\mu_1(B) = \sum_k \alpha_k r_{1,k} \delta_{x_{1,k}}(B) = \int_{X_0 \times \mathbb{R}_+ \times B \times \mathbb{R}_+} r_1 \, d\boldsymbol{\alpha}(x_0, r_0; x_1, r_1)$$

$$= \mathfrak{h}_1 \boldsymbol{\alpha}(B)$$

$\mu_0 = \mathfrak{h}_0 \boldsymbol{\alpha} = \pi_\sharp^{X_0}(r_0 \boldsymbol{\alpha})$, $\quad \mu_1 = \mathfrak{h}_1 \boldsymbol{\alpha} = \pi_\sharp^{X_1}(r_1 \boldsymbol{\alpha})$ 1-homogeneous marginals of $\boldsymbol{\alpha}$

Representation

We introduce the set of plans with homogeneous marginals μ_0, μ_1:

$$\mathfrak{H}(\mu_0, \mu_1) := \Big\{ \boldsymbol{\alpha} \in \mathcal{P}(X_0 \times \mathbb{R}_+ \times X_1 \times \mathbb{R}_+) : \\ \mathfrak{h}_0 \boldsymbol{\alpha} = \pi^{X_0}_{\sharp}(r_0 \boldsymbol{\alpha}) = \mu_0,\ \mathfrak{h}_1 \boldsymbol{\alpha} = \pi^{X_1}_{\sharp}(r_1 \boldsymbol{\alpha}) = \mu_1 \Big\}$$

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$$\boxed{\mathsf{UOT}(\mu_0, \mu_1) = \min \left\{ \int \mathsf{h}(x_0, r_0; x_1, r_1) \, \mathrm{d}\boldsymbol{\alpha} : \boldsymbol{\alpha} \in \mathfrak{H}(\mu_0, \mu_1) \right\}}$$

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It is possible to check that UOT **is convex, l.s.c., and it is dominated by** $\mathrm{UOT}_{\mathrm{Dirac}}$, so that

$$\mathrm{UOT}(\mu_0,\mu_1) \leqslant \mathrm{UOT}^{**}_{\mathrm{Dirac}}(\mu_0,\mu_1)$$

On the other hand it is also immediate to check that

$$\mathrm{UOT}(\mu_0,\mu_1) \geqslant \mathrm{UOT}^{**}_{\mathrm{Dirac}}(\mu_0,\mu_1),$$

Primal-dual equivalence of Unbalanced Optimal Transport

$$\mathrm{UOT}(\mu_0,\mu_1) = \mathrm{UOT}^{**}_{\mathrm{Dirac}}(\mu_0,\mu_1) = \sup\Big\{ \int \phi_0\,d\mu_0 + \int \phi_1\,d\mu_1 : (\phi_0,\phi_1) \in K[h] \Big\},$$

$$K[h] = \Big\{ (\phi_0,\phi_1) \in C_b(X_0) \times C_b(X_1) : r_0\phi_0(x_0) + r_1\phi_1(x_1) \leqslant h(x_0,r_0;x_1,r_1) \Big\}.$$

The link with Optimal Transport in the cone space

Consider the space $\mathfrak{C}[X_i] = (X_i \times \mathbb{R}_+)/\sim$ and the cost functional h. It induces the OT problem

$$\mathrm{OT}_h(\alpha_0, \alpha_1) = \min \left\{ \int h \, d\boldsymbol{\alpha} : \boldsymbol{\alpha} \in \Gamma(\alpha_1, \alpha_2) \right\}.$$

We have

Optimal transport formulation via homogeneous marginals

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How to choose interesting costs h? We discuss the particular case of the hellinger-Kantorovich metric, induced by the natural cone distance on $\mathfrak{C}[\mathbb{R}^d]$.

Outline

Let $\mu \in C^0([0,1];\mathcal{M}(\mathbb{R}^d))$, $(\boldsymbol{v}, w) : \mathbb{R}^d \times (0,1) \to \mathbb{R}^{d+1}$ be a Borel vector field satisfying

$$\int_0^1 \int \left(|\boldsymbol{v}_t(x)|^2 + w_t^2(x) \right) d\mu_t(x)\, dt < \infty.$$

Continuity equation with reaction governed by the field $(\boldsymbol{v}, w)$ if

$$\partial_t \mu_t + \nabla \cdot (\boldsymbol{v}_t \mu_t) = 2 w_t \mu_t \quad \text{in } \mathscr{D}'(\mathbb{R}^d \times (0,1)) \tag{CER}$$

The dynamic perspective

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The Hellinger-Kantorovich distance via dynamic interpolation

$$\mathsf{HK}^2(\mu_0, \mu_1) = \min \Big\{ \int_0^1 \int \Big(|\boldsymbol{v}_t|^2 + |w_t|^2\Big)\, d\mu_t\, dt : \mu \in C([0,1];\mathcal{M}(\mathbb{R}^d)),$$
$$\partial_t \mu_t + \nabla \cdot (\boldsymbol{v}_t \mu_t) = 2 w_t \mu_t, \quad \mu_{t=i} = \mu_i \Big\}.$$

This approach has been independently proposed by

Kondratiev, Monsaingeon, Vorotnikov and Chizat, Peyré, Vialard, Schmitzer.

The distances between two Dirac masses

Suppose that $\mu_i = r_i^2 \delta_{x_i}$; if we look for $\mu_t := r^2(t)\delta_{x(t)}$

$$\partial_t \mu_t + \nabla \cdot (\mu_t \boldsymbol{v}_t) = 2 w_t \mu_t, \quad \boldsymbol{v}_t(x(t)) = \dot{x}(t), \quad w_t(x(t)) = \dot{r}(t)/r(t)$$

We can compute

$$\mathsf{HK}^2(r_0^2 \delta_{x_0}, r_1^2 \delta_{x_1}) = \min \left\{ \int_0^1 \left(r^2(t) |\dot{x}(t)|^2 + |\dot{r}(t)|^2 \right) dt : \right.$$
$$\left. (x, r) : [0,1] \to \mathbb{R}^d \times \mathbb{R}_+, \ (x(i), r(i)) = (x_i, r_i) \ i = 0, 1 \right\}$$

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HK is associated to the cone distance:

$$\mathsf{d}^2_{\mathfrak{C}}((x_0,r_0),(x_1,r_1)) = r_0^2 + r_1^2 - 2 r_0 r_1 \cos_\pi(|x_1 - x_0|)$$

where $\cos_\alpha(r) = \cos(r \wedge \alpha)$. $\mathsf{d}_{\mathfrak{C}}((x_0,r_0),(x_1,r_1))$ is a **length distance**.

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Truncation effect: when $|x_0 - x_1| \geqslant \pi/2$ a better competitor is provided by $\mu_t := [(1-t)r_0]^2\delta_{x_0} + (t r_1)^2 \delta_{x_1}$ and we have

$$\mathsf{HK}^2(r_0^2\delta_{x_0}, r_1^2\delta_{x_1}) = r_0^2 + r_1^2.$$

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$$\partial_t\mu_t + \nabla\cdot(\mu_t v_t) = 2w_t\mu_t, \quad v_t(x(t)) = \dot{x}(t), \quad w_t(x(t)) = \dot{r}(t)/r(t)$$

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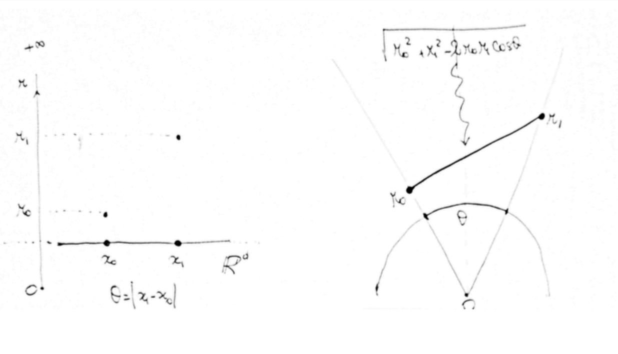
The Cone space

Cone metric: $d^2_{\mathfrak{C}}((x_0, r_0), (x_1, r_1)) = r_0^2 + r_1^2 - 2r_0 r_1 \cos_\pi(|x_1 - x_0|)$

Cone space: **identify all the points $(x, 0)$ with the vertex $\mathfrak{o}$.**

$$\mathfrak{C} := \big(\mathbb{R}^d \times [0, \infty)\big)/\sim, \quad (x', r') \sim (x'', r'') \quad \Leftrightarrow \quad \begin{cases} x' = x'',\ r' = r'' \neq 0, \\ r' = r'' = 0 \end{cases}$$



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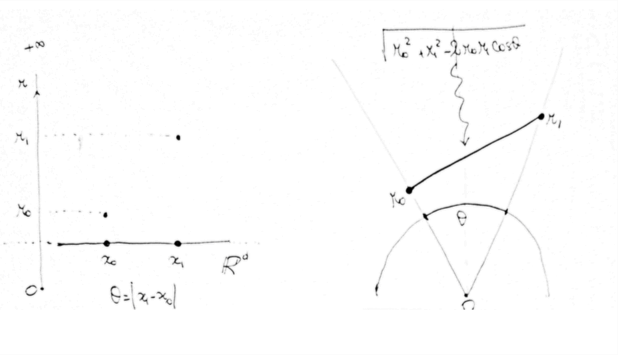

$\mathfrak{C} \setminus \{\mathfrak{o}\}$ can be considered as a **smooth Riemannian manifold** with metric

Unbalanced transport: the link with the relaxation viewpoint

HK^2 is a **convex and subadditive functional**

We introduce a function h : $(\mathbb{R}^d \times \mathbb{R}_+)^2 \to [0, +\infty)$ which characterizes the cost of connecting two Dirac measures with possibly different mass:

$$
\begin{aligned}
h(x_0, r_0; x_1, r_1) &:= \mathrm{HK}^2(r_0\delta_{x_0}, r_1\delta_{x_1}) = d_{\mathfrak{C}}^2((x_0, \sqrt{r_0}), (x_1, \sqrt{r_1})) \\
&= r_0 + r_1 - 2\sqrt{r_0 r_1}\cos_{\pi/2}(|x_1 - x_0|) \quad x_i \in X_i,\ r_i \geqslant 0
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$$(r_0, r_1) \mapsto h(x_0, r_0; x_1, r_1) \quad \text{is } \textbf{positively 1-homogeneous and convex}$$

thanks to the truncation ($-\cos_{\pi/2} \leqslant 0$). Define $\mathrm{UOT}_{\mathrm{Dirac}}(\mu_0, \mu_1) := \mathrm{HK}^2(\mu_0, \mu_1)$ if $\mu_i = r_i\delta_{x_i}$, $+\infty$ otherwise.

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Theorem

HK^2 *is the* Γ*-relaxation of* $\mathrm{UOT}_{\mathrm{Dirac}}$*: the* ***largest convex and lower semicontinuous functional*** *defined in* $\mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d) \to [0, +\infty]$ *dominated by* $\mathrm{UOT}_{\mathrm{Dirac}}$*:*

$$\widehat{\mathrm{UOT}}\ \textit{convex, l.s.c.},\ \widehat{\mathrm{UOT}} \leqslant \mathrm{UOT}_{\mathrm{Dirac}} \quad \Rightarrow \quad \widehat{\mathrm{UOT}} \leqslant \mathrm{HK}^2.$$

$$\mathfrak{H}(\mu_0, \mu_1) := \Big\{ \boldsymbol{\alpha} \in \mathcal{P}(X_0 \times \mathbb{R}_+ \times X_1 \times \mathbb{R}_+) :$$
$$\mathfrak{h}^0 \boldsymbol{\alpha} = \pi_\sharp^{X_0}(r_0^2 \boldsymbol{\alpha}) = \mu_0,\ \mathfrak{h}^1 \boldsymbol{\alpha} = \pi_\sharp^{X_1}(r_1^2 \boldsymbol{\alpha}) = \mu_1 \Big\}$$

Representation

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Primal formulation

$$\begin{aligned} \mathsf{HK}^2(\mu_0, \mu_1) &= \min \left\{ \int h(x_0, r_0^2; x_1, r_1^2)\, d\boldsymbol{\alpha} : \boldsymbol{\alpha} \in \mathfrak{H}(\mu_0, \mu_1) \right\} \\ &= \min \left\{ \int d_{\mathfrak{C}}^2((x_0, r_0), (x_1, r_1))\, d\boldsymbol{\alpha} : \boldsymbol{\alpha} \in \mathfrak{H}(\mu_0, \mu_1) \right\} \end{aligned}$$

Transport-growth pairs

We can represent $\boldsymbol{\alpha} \in \mathfrak{H}(\mu_0, \mu_1)$ as $\boldsymbol{\alpha} = ((\mathbf{T}_0, q_0), (\mathbf{T}_1, q_1))_\sharp \boldsymbol{\lambda}$ where $\boldsymbol{\lambda} \in \mathcal{M}(Y)$, Y is some Polish space, and $(\mathbf{T}_i, q_i) : Y \to \mathbb{R}^d \times \mathbb{R}_+$ with $q_i \in L^2(\boldsymbol{\lambda})$.

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We say that $(\mathbf{T}_i, q_i)$ is a **transport-growth pair.** $(\mathbf{T}, q)$ acts on λ according to this rule:

$$(\mathbf{T}, q)_\star \lambda := \mathbf{T}_\sharp (q^2 \lambda) = \mathfrak{h}((\mathbf{T}, q)_\sharp \lambda),$$

$$\boxed{\begin{aligned} \mathsf{HK}^2(\mu_0, \mu_1) = \min \Big\{ \int_{Y \times Y} \Big(q_0^2 + q_1^2 - 2 q_0 q_1 \cos_{\pi/2} |\mathbf{T}_0 - \mathbf{T}_1| \Big) \, d\lambda \,\Big|\, \lambda \in \mathcal{M}(Y), \\ Y \text{ Polish}, (\mathbf{T}_i, q_i) : Y \to \mathbb{R}^d \times \mathbb{R}_+, \ \mu_i := (\mathbf{T}_i, q_i)_\star \lambda \Big\}; \end{aligned}}$$

moreover, it is not restrictive to choose $Y = \mathfrak{C}[\mathbb{R}^d] \times \mathfrak{C}[\mathbb{R}^d]$.

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Problem (Monge formulation of HK)

Given $\mu_0, \mu_1 \in \mathcal{M}(\mathbb{R}^d)$ find an optimal transport-growth pair $(\mathbf{T}, q) : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}_+$ minimizing the cost

$$\mathscr{M}(\mathbf{T}, q; \mu_0) := \int \Big(1 + q^2(x) - 2q(x) \cos_{\pi/2} |\mathbf{T}(x) - x|\Big)\, \mathrm{d}\mu_0(x) \tag{1}$$

among all the transport-growth maps satisfying $(\mathbf{T}, q)_\star \mu_0 = \mu_1$

Duality with the conical Hamilton-Jacobi equation

If

$$\partial_t \xi_t + \frac{1}{2}|D\xi_t|^2 + 2\xi_t^2(x) \leqslant 0 \tag{CHJ}$$

and

$$\partial_t \mu_t + \nabla \cdot (\boldsymbol{v}_t \mu_t) = 2w_t \mu_t$$

then

$$\int \xi_1 \, d\mu_1 - \int \xi_0 \, d\mu_0 \leqslant \frac{1}{2} \int_0^1 \int \left(|\boldsymbol{v}_t|^2 + w_t^2 \right) d\mu_t \, dt.$$

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HK in duality with subsolutions to the conical Hamilton-Jacobi equations

$$\frac{1}{2}\mathsf{HK}^2(\mu_0, \mu_1) = \sup \Big\{ \int \xi_1 \, d\mu_1 - \int \xi_0 \, d\mu_0 : \xi \in C^1([0,1]; \mathrm{Lip}_b(\mathbb{R}^d)) \\ \partial_t \xi_t + \frac{1}{2}|D\xi_t|^2 + 2\xi_t^2 \leqslant 0 \Big\}.$$

Conical Hopf-Lax representation formula

Given $\xi_0 \in \mathrm{Lip}_b(\mathbb{R}^d)$ with $\xi_0 > -1/2$, the viscosity solution (or the maximal subsolution) of the conical Hamilton Jacobi equation

$$\partial_t \xi_t + \frac{1}{2}|D\xi_t|^2 + 2\xi_t^2 = 0 \tag{CHJ}$$

is given by the **conical Hopf-Lax semigroup** (cf. BARRON-JENSEN-LIU for different representation formulae)

$$\boxed{\mathscr{P}_t \xi(x) := \inf_y \frac{1}{2t}\left[1 - \frac{\cos^2_{\pi/2}(|y-x|)}{1+2t\xi(x)}\right]} \tag{CHL}$$

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Conical Hopf-Lax representation for HK

$$\frac{1}{2}\mathsf{HK}^2(\mu_0,\mu_1) = \sup\left\{\int \xi_1 \, d\mu_1 - \int \xi_0 \, d\mu_0 : \xi_1 = \mathscr{P}_1\xi_0\right\}$$

Formally, if ξ is a solution of

$$\partial_t \xi_t + \frac{1}{2}|D\xi_t|^2 + 2\xi_t^2 \leqslant 0 \tag{CHJ}$$

then $\boxed{\zeta_t(x,r) := \xi_t(x) r^2}$ is a solution of

$$\partial_t \zeta_t + \frac{1}{2}|D_{\mathfrak{C}} \zeta_t|^2 \leqslant 0 \tag{HJ}$$

since

$$\frac{1}{2}|D_{\mathfrak{C}}\zeta|^2 = \frac{1}{2}\mathfrak{g}^*(D_x\zeta, \partial_r\zeta) = \frac{1}{2}\left(\frac{1}{r^2}|D_x\zeta|^2 + (\partial_r\zeta)^2\right) = \left(\frac{1}{2}|D\xi_t|^2 + 2\xi_t^2\right)r^2$$

Conical lift of the Hopf-Lax formula

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The Hopf-Lax semigroup in $\mathfrak{C}$

$$\begin{aligned}\mathscr{Q}_t^{\mathfrak{C}}\zeta(x, r) &= \min_{y,s} \zeta(y, s) + \frac{1}{2t} d_{\mathfrak{C}}^2((x, r), (y, s)) \\ &= \min_{y,s} \xi(y) s^2 + \frac{1}{2t}\Big(r^2 + s^2 - 2rs\cos(|x-y|_\pi)\Big)\end{aligned}$$

yields

$$\mathscr{Q}_t^{\mathfrak{C}}\zeta(x, r) = \xi_t(x) r^2, \quad \xi_t = \mathscr{P}_t \xi.$$

Dual formulation (II)

Change of variable: $\varphi_1 := -\frac{1}{2}\log(1-2\xi_1)$, $\varphi_0 := \frac{1}{2}\log(1+2\xi_0)$

$$2\xi_1(y) \leqslant 1 - \frac{\cos^2_{\pi/2}(|y-x|)}{1+2\xi_0(x)} \quad \Leftrightarrow \quad \varphi_1(y) - \varphi_0(x) \leqslant \ell(y-x),$$

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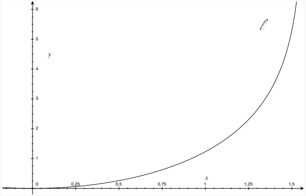

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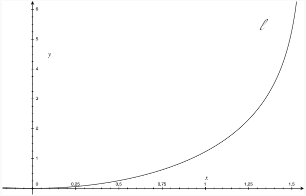

Dual Kantorovich formulation

$$\frac{1}{2}\mathsf{HK}^2(\mu_0,\mu_1) = \sup\left\{ \int \frac{1}{2}(1-e^{-2\varphi_1})\,d\mu_1 - \int \frac{1}{2}(e^{2\varphi_0}-1)\,d\mu_0 : \right.$$
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Primal formulation: Logarithmic Entropy-Transport problem

The **Legendre conjugate** of $\mathrm{G}(\varphi) := \frac{1}{2}\left(\mathrm{e}^{2\varphi} - 1\right)$ is

$$\mathrm{G}^*(s) = \frac{1}{2}\mathrm{LE}(s), \qquad \mathrm{LE}(s) := s \log s - (s-1)$$

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Logarithmic Entropy-Transport (LET) formulation

$$\mathsf{LET}(\mu_0, \mu_1) = \min_{\boldsymbol{\gamma}\in\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)} \left(\mathscr{E}(\gamma_0|\mu_0) + \mathscr{E}(\gamma_1|\mu_1) + 2\int \ell(y-x)\,d\boldsymbol{\gamma}(x,y)\right)$$

where $\ell(\mathbf{r}) = \frac{1}{2}\log(1 + \tan^2_{\pi/2}(|\mathbf{r}|))$.

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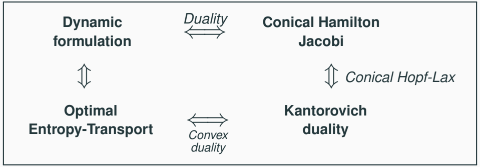
Four equivalent formulations for HK



Four equivalent formulations for HK

Dynamic formulation	$\overset{\text{Duality}}{\Longleftrightarrow}$	**Conical Hamilton Jacobi**
$\Updownarrow$		$\Updownarrow$ *Conical Hopf-Lax*
Optimal Entropy-Transport	$\underset{\text{Convex duality}}{\Longleftrightarrow}$	**Kantorovich duality**

$$\mathsf{HK}^2(\mu_0,\mu_1) = \min\Big\{\int_0^1\int\Big(|v_t|^2+|w_t|^2\Big)\,d\mu_t\,dt : \mu\in C([0,1];\mathcal{M}(\mathbb{R}^d)),$$

$$\partial_t\mu_t+\nabla\cdot(v_t\mu_t)=2w_t\mu_t,\quad \mu_{t=i}=\mu_i\Big\} \tag{CER}$$

$$= 2\sup\Big\{\int\xi_1\,d\mu_1-\int\xi_0\,d\mu_0 : \xi\in C^1([0,1];\mathrm{Lip}_b(\mathbb{R}^d))$$

$$\partial_t\xi_t+\frac{1}{2}|D\xi_t|^2+2\xi_t^2\leqslant 0\Big\} \tag{CHJ}$$

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$$= \min_{\gamma}\mathscr{E}(\gamma_0|\mu_0)+\mathscr{E}(\gamma_1|\mu_1)+2\int\ell(x,y)\,d\boldsymbol{\gamma}(x,y). \tag{LET}$$

Outline

Important properties

- $(\mathcal{M}(\mathbb{R}^d), \mathsf{HK})$ is a **complete and separable metric space** if X is complete and separable; the induced topology coincides with the topology of weak convergence (no bounds on moments are required).

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Problem

Characterize geodesics and study the convexity properties of integral functionals.

In particular, we want to prove that power-like entropies

$$\mathscr{E}_\alpha(\mu) := \int c^\alpha \, dx, \qquad \mu = c\mathscr{L}^d$$

are **geodesically convex** if $\alpha \geqslant 1$ (reinforced McCann condition).

The $\pi/2$ treshold and HK geodesics between Dirac masses

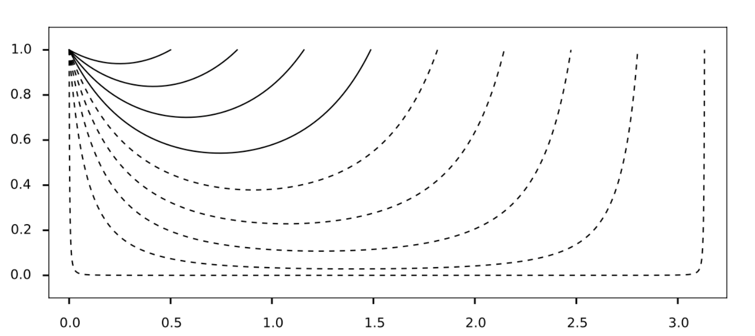
$\mu_0 = r_0^2\delta_{x_0}$, $\mu_1 = r_1^2\delta_{x_1}$, $|x_1 - x_0| \in [0,\pi]$, $\mu_t := r_t\delta_{x_t}$ geodesic.

Initial velocities $(u, \boldsymbol{v}) \in \mathbb{R} \times \mathbb{R}^d$

$$u := \frac{r_1}{r_0}\cos(|x_1 - x_0|) - 1 \qquad \boldsymbol{v} := \frac{r_1}{r_0}\underline{\sin}(x_1 - x_0), \qquad \underline{\sin}(\boldsymbol{w}) := \sin(|\boldsymbol{w}|)\frac{\boldsymbol{w}}{|\boldsymbol{w}|}$$

curve:

$$r_t := r_0\Big((1+tu)^2 + t^2|\boldsymbol{v}|^2\Big)^{1/2}, \qquad x_t := x_0 + \underline{\arctan}\Big(\frac{t\boldsymbol{v}}{1+tu}\Big)$$



Regularity of optimal potentials for the LET formulation

Theorem

For every $\mu_0, \mu_1 \in \mathcal{M}(\mathbb{R}^d)$ *there exists a pair of optimal potentials* (φ_0, φ_1) *such that* $\varphi_1(y) - \varphi_0(x) \leqslant 2\ell(y - x)$ *and*

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The Optimal potential φ_0 *is locally semiconcave outside a closed* $(d-1)$*-rectifiable set.*

Regularity of optimal potentials for the LET formulation

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For every $\mu_0, \mu_1 \in \mathcal{M}(\mathbb{R}^d)$ *there exists a pair of optimal potentials* (φ_0, φ_1) *such that* $\varphi_1(y) - \varphi_0(x) \leqslant 2\ell(y-x)$ *and*

$$\mathsf{HK}^2(\mu_0, \mu_1) = \int (1 - e^{-2\varphi_1})\, d\mu_1 - \int (e^{2\varphi_0} - 1)\, d\mu_0.$$

The Optimal potential φ_0 *is locally semiconcave outside a closed* $(d-1)$*-rectifiable set.*
When $\mu_0 \ll \mathscr{L}^d$ *and* $\mu_1\left\{y \in \mathbb{R}^d : d(y, \operatorname{supp}(\mu_0)) \geqslant \pi/2\right\} = 0$*, then Monge formulation has a unique solution* (T, q) *such that* $(\mathsf{T}, q)_*\mu_0 = \mu_1$ *and*

$$\underline{\tan}(\mathsf{T}(x) - x) = \nabla\varphi_0(x), \quad q^2(x) = (e^{2\varphi_0(x)})^2 + \frac{1}{4}|\nabla e^{2\varphi_0}(x)|^2$$

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After the transformation $\xi_0 := \frac{1}{2}\left(e^{2\varphi_0} - 1\right)$ we can identify

$$\boxed{\mathbf{T}(x) = x + \underline{\mathsf{arctan}}\left(\frac{\nabla\xi_0}{1+2\xi_0}\right), \qquad q^2 = (1+2\xi_0)^2 + |\nabla\xi_0|^2}$$

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$$\boxed{\mathsf{HK}^2(\mu_0, \mu_1) = \mathscr{M}(\mathsf{T}, q; \mu_0) = \int_{\mathbb{R}^d} \Big(4\xi_0^2 + |\nabla\xi_0|^2\Big)\, d\mu_0}$$

Tangent space: $\mathrm{Tan}_{\mu_0}\mathcal{M}(\mathbb{R}^d) = H^{1,2}(\mathbb{R}^d, \mu_0)$.

Geodesics

Recalling

$$\mathsf{T}(x) = x + \underline{\arctan}\left(\frac{\nabla \xi_0}{1+2\xi_0}\right), \qquad q^2 = (1+2\xi_0)^2 + |\nabla \xi_0|^2$$

the geodesic interpolations can be obtained by rescaling $\xi_0 \rightsquigarrow t\xi_0$, $t \in [0, 1]$:

$$\boxed{\mathsf{T}_{0\to t}(x) := x + \underline{\arctan}\left(\frac{t\nabla \xi_0}{1+2t\xi_0(x)}\right), \quad q_{0\to t}^2(x) := (1+2t\xi_0(x))^2 + t^2|\nabla \xi_0(x)|^2}$$

They provide an explicit characterization of the **unique HK geodesic connecting μ_0 to μ_1:**

$$\boxed{\mu_t = (\mathsf{T}_{0\to t}, q_{0\to t})_* \mu_0, \quad \mu_t = c_t \mathscr{L}^d, \quad c_t(\mathsf{T}_{0\to t}(x)) = c_0(x)\frac{q_{0\to t}^2(x)}{\det \mathsf{D}\mathsf{T}_{0\to t}(x)}}$$

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Simplifying assumption: μ_0, μ_1 have compact support, $\mathsf{supp}(\mu_1) \subset B_{\pi/2}(\mathsf{supp}(\mu_0))$, $\mathsf{supp}(\mu_0) \subset B_{\pi/2}(\mathsf{supp}(\mu_1))$.

Optimal potentials φ_0 and ξ_0 are semiconvex, φ_1 and ξ_1 are semiconcave, all the functions are globally Lipschitz and for suitable constants $a, b \in \mathbb{R}$

$$-\frac{1}{2} < -a \leqslant \xi_0(x) \leqslant b, \quad -b \leqslant \xi_1(y) \leqslant a < \frac{1}{2}.$$

Dynamic optimality conditions for geodesics

Theorem (Formal)

A continuous curve $(\mu)_{t\in[0,1]}$ *is a geodesic if and only if there exists a curve* $(\xi_t)_{t\in[0,1]}$ *such that*

$$\begin{cases} \partial_t \mu_t + \nabla \cdot (\mu_t \boldsymbol{v}_t) = 2 w_t \mu_t \\ \partial_t \xi_t + \frac{1}{2} |\nabla \xi_t|^2 + 2\xi_t^2 \leqslant 0 \\ \partial_t \xi_t + \frac{1}{2} |\nabla \xi_t|^2 + 2\xi_t^2 = 0 \quad \textit{on the support of } \mu, \\ \boldsymbol{v}_t = \nabla \xi_t \\ w_t = 2\xi_t \end{cases}$$

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Characteristic flow: fix $s\in(0,1)$ $\mathbf{T}(t,\cdot) := \mathbf{T}_{s\to t}(\cdot)$, $q(t,\cdot) := q_{s\to t}(\cdot)$,

$$\begin{cases} \dot{\mathbf{T}}(t,x) = \nabla\xi_t(\mathbf{T}(t,x)) \\ \dot{q}(t,x) = 4\xi_t(\mathbf{T}(t,x))q(t,x) \\ \mathbf{T}(s,x) = x, \\ q(s,x) = 1. \end{cases}$$

Formal computations

$$\partial_t \xi_t + \frac{1}{2}|\nabla \xi_t|^2 + 2\xi_t^2 = 0.$$

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Second order relations

$$\ddot{\mathbf{T}}(t) = 4\xi_t \nabla \xi_t(\mathbf{T}(t))$$

$$\ddot{q}(t) = |\nabla \xi_t(\mathbf{T}(t))|^2 q(t)$$

$$\ddot{B}(t) = -4\Big(\nabla \xi_t \otimes \nabla \xi_t + \xi_t D^2\xi_t\Big) \circ \mathbf{T}(t) \cdot B(t)$$

$$\ddot{\delta}(t) = \Big((\Delta\xi_t)^2 - |D^2\xi_t|^2 - 4|\nabla \xi_t|^2 - 4\xi_t\Delta\xi_t\Big) \circ \mathbf{T}(t) \cdot \delta(t).$$

Structural second order estimates for the densities

$\mu_t = c(t, \cdot)\mathscr{L}^d$ with

$$c(t) = \frac{q^2(t)}{\delta(t)} = \frac{q^{d+2}(t)}{q^d(t)\delta(t)} = \frac{q^{d+2}(t)}{\rho^d(t)}, \qquad \boxed{\rho(t) := q(t)\delta^{1/d}(t)}$$

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since the previous identities yield

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Theorem

*The density $c(t,\cdot)$ is **convex along the characteristics:***

$$\frac{\ddot c}{c} \geqslant 6\frac{\ddot q}{q} \geqslant 0.$$

*The functional $\mu \mapsto \|d\mu/d\mathscr{L}^d\|_{L^\infty}$ is **geodesically convex.***

Application: geodesic convexity of integral functionals

Consider a functional

$$\mathscr{E}(\mu) := \int E(c(x))\,dx, \quad c = \frac{d\mu}{d\mathscr{L}^d}$$

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More generally, we set $\varepsilon_0(c) := E(c)$, $\varepsilon_1(c) := cE'(c)$, $\varepsilon_2(c) := c^2E''(c)$.

McCann condition:

$$\varepsilon_2(c) \geqslant \left(1 - \frac{1}{d}\right)(\varepsilon_1(c) - \varepsilon_0(c)) \geqslant 0 \quad \Leftrightarrow \quad r^d E(r^{-d}) \text{ convex, nonincreasing.}$$

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Theorem

$\mathscr{E}$ *is geodesically convex w.r.t.* HK *if and only if*

$$G(c) := \begin{pmatrix} \varepsilon_2(c) - \frac{d-1}{d}(\varepsilon_1(c) - \varepsilon_0(c)) & \varepsilon_2(c) - \frac{1}{2}(\varepsilon_1(c) - \varepsilon_0(c)) \\ \varepsilon_2(c) - \frac{1}{2}(\varepsilon_1(c) - \varepsilon_0(c)) & \varepsilon_2(c) + \frac{1}{2}\varepsilon_1(c) \end{pmatrix} \geqslant 0, \quad \varepsilon_1 \geqslant \varepsilon_0.$$

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Define

$$N(\rho, q) := \left(\frac{\rho}{q}\right)^d E\left(\frac{q^{d+2}}{\rho^d}\right)$$

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Main examples: the power functions $E(c) := c^p$ are convex if $p \geqslant 1$.

In dimension $d = 2$ also $E(c) = -\sqrt{c}$ is convex.

In dimension $d = 1$ all the power functions $E(c) = -c^p$, $p \in [1/3, 1/2]$ induces convex functionals.

Outline

A rigorous proof: regularity of CHL solutions (I)

Conical Hopf-Lax representation formula:

$$\mathscr{P}_t \xi(x) := \inf_y \frac{1}{2t}\left[1 - \frac{\cos^2_{\pi/2}(|y-x|)}{1+2t\xi(x)}\right] \tag{CHL}$$

It is useful to introduce the reverse evolution (Villani '09)

$$\mathscr{R}_t \bar{\xi}(x) := -\mathscr{P}_{1-t}(-\bar{\xi})(x) = \sup_y \frac{1}{2(1-t)}\left[\frac{\cos^2_{\pi/2}(|y-x|)}{1-2(1-t)\bar{\xi}(x)} - 1\right] \tag{RCHL}$$

If $\bar{\xi}_1 : \mathbb{R}^d \to [-b, a]$ *with* $-\infty < -b < a < 1/2$ *then the functions* $\bar{\xi}_t := \mathscr{R}_t \xi_1(x)$ *are globally bounded, Lipschitz and semiconvex,* $\xi_t = \mathscr{P}_{t-s}\xi_s$, $\xi_1 < 1/2$.

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Regulairty of CHL (II)

Theorem (Differentiability on the contact set)

If $\bar{\xi}_1 = \xi_1 = \mathscr{P}_1(\xi_0)$, $\xi_0 = \mathscr{R}_1 \xi_1$ *then* $\xi_t \geqslant \bar{\xi}_t$ *and the* ***contact set***

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the map $\mathsf{T}_{s\to t}$ *is* ***Lipschitz****, it satisfies* $\boxed{\mathsf{T}_{s\to t}(\Xi_s) = \Xi_t}$ *and*

the concatenation property $\boxed{\mathsf{T}_{t_1\to t_2} \circ \mathsf{T}_{t_0\to t_1} = \mathsf{T}_{t_0\to t_2}}$

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g_t *is* ***Lipschitz*** *in* Ξ_t *and for every* $s, t \in [0,1]$ *defining*

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the map $\mathsf{T}_{s\to t}$ *is* ***Lipschitz****, it satisfies* $\boxed{\mathsf{T}_{s\to t}(\Xi_s) = \Xi_t}$ *and*

the concatenation property $\boxed{\mathsf{T}_{t_1\to t_2}\circ \mathsf{T}_{t_0\to t_1} = \mathsf{T}_{t_0\to t_2}}$

Setting

$$q^2_{s\to t}(x) := \big(1+2(t-s)\xi_s(x)\big)^2 + (t-s)^2|\mathsf{g}_s(x)|^2$$

we have $\boxed{q_{t_1\to t_2}\circ \mathsf{T}_{t_0\to t_1}\cdot q_{t_0\to t_1} = q_{t_0\to t_2}}$

Theorem

For every $s \in (0, 1)$ *and* $t \in [0, 1]$ *the transport-growth pair* $(T_{s \to t}, q_{s \to t})$ *is the* ***unique solution to the Monge formulation*** *for the* HK *problem between* μ_s *and* μ_t.

Nonbranching and restrictions

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In particular, if for given μ_0, μ_1, μ_s

$$\mathsf{HK}(\mu_0, \mu_s) = s\mathsf{HK}(\mu_0, \mu_1), \qquad \mathsf{HK}(\mu_s, \mu_1) = (1-s)\mathsf{HK}(\mu_0, \mu_1)$$

then there exists a unique geodesic $\mu : [0,1] \to \mathcal{M}(\mathbb{R}^d)$ connecting μ_0 to μ_1 such that $\mu(s) = \mu_s$.

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If $\mu_s \ll \mathscr{L}^d$ *then* $\mu_t \ll \mathscr{L}^d$ *for every* $t \in (0,1)$*.*

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If $\mu_s \ll \mathscr{L}^d$ then $\mu_t \ll \mathscr{L}^d$ for every $t \in (0,1)$.

If $\mathrm{supp}(\nu_s) \subset \mathrm{supp}(\mu_s)$ then $\nu_t := (T_{s\to t}, q_{s\to t})_* \nu_s$ is a geodesic.

Second order regularity of CHL (III)

Let $\mathfrak{D}_s \subset \Xi_s$ the **set of points of density** 1 **where** g_s **is differentiable**.

Theorem

$A_s := \mathrm{D}\mathbf{g}_s$ *is* ***symmetric****.* ξ_s *has a* ***second order Taylor expansion*** *in terms of* g_s *and* A_s*. We thus can set* $\mathbf{g}_s = \nabla\xi_s$, $B_s = \mathrm{D}\nabla\xi_s = \mathrm{D}^2\xi_s$ *in* $\mathfrak{D}_s$*.*

$$
\begin{cases}
\ddot{\mathbf{T}}(t) = 4\xi_t \nabla\xi_t(\mathbf{T}(t)) \\
\ddot{q}(t) = |\nabla\xi_t(\mathbf{T}(t))|^2 q(t) \\
\ddot{B}(t) = -4\Big(\nabla\xi_t \otimes \nabla\xi_t + \xi_t \mathrm{D}^2\xi_t\Big) \circ \mathbf{T}(t) \cdot B(t) \\
\ddot{\delta}(t) = \Big((\Delta\xi_t)^2 - |\mathrm{D}^2\xi_t|^2 - 4|\nabla\xi_t|^2 - 4\xi_t\Delta\xi_t\Big) \circ \mathbf{T}(t) \cdot \delta(t).
\end{cases}
$$

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$A_s := Dg_s$ is ***symmetric****. ξ_s has a* ***second order Taylor expansion*** *in terms of g_s and A_s. We thus can set $g_s = \nabla \xi_s$, $B_s = D\nabla\xi_s = D^2\xi_s$ in $\mathfrak{D}_s$.*

If $\mu_s \ll \mathscr{L}^d$ then $\boxed{\mu_s(\Xi_s \setminus \mathfrak{D}_s) = 0}$.

$$\begin{cases} \ddot{T}(t) = 4\xi_t \nabla\xi_t(T(t)) \\ \ddot{q}(t) = |\nabla\xi_t(T(t))|^2 q(t) \\ \ddot{B}(t) = -4\Big(\nabla\xi_t \otimes \nabla\xi_t + \xi_t D^2\xi_t\Big) \circ T(t) \cdot B(t) \\ \ddot{\delta}(t) = \Big((\Delta\xi_t)^2 - |D^2\xi_t|^2 - 4|\nabla\xi_t|^2 - 4\xi_t\Delta\xi_t\Big) \circ T(t) \cdot \delta(t). \end{cases}$$

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$A_s := Dg_s$ *is* ***symmetric***. ξ_s *has a* ***second order Taylor expansion*** *in terms of* g_s *and* A_s. *We thus can set* $g_s = \nabla\xi_s, B_s = D\nabla\xi_s = D^2\xi_s$ *in* $\mathfrak{D}_s$.

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$\mathbf{T}_{s\to t}$ *is* ***differentiable*** *in* $\mathfrak{D}_s$ *and* $\mathbf{T}_{s\to t}(\mathfrak{D}_s) = \mathfrak{D}_t$.
The maps $\mathbf{T}(t) := \mathbf{T}_{s\to t}$, $B(t,\cdot) := D\mathbf{T}(t,\cdot)$, $\delta(t,\cdot) := \det B(t,\cdot)$ *are analytic in time and satisfy the characteristic systems of ODE.*

$$\begin{cases} \ddot{\mathbf{T}}(t) = 4\xi_t \nabla\xi_t(\mathbf{T}(t)) \\ \ddot{q}(t) = |\nabla\xi_t(\mathbf{T}(t))|^2 q(t) \\ \ddot{B}(t) = -4\Big(\nabla\xi_t \otimes \nabla\xi_t + \xi_t D^2\xi_t\Big) \circ \mathbf{T}(t) \cdot B(t) \\ \ddot{\delta}(t) = \Big((\Delta\xi_t)^2 - |D^2\xi_t|^2 - 4|\nabla\xi_t|^2 - 4\xi_t\Delta\xi_t\Big) \circ \mathbf{T}(t) \cdot \delta(t). \end{cases}$$

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