# Modular forms and an explicit Chebotarev variant of the Brun-Titchmarsh theorem
# Comparative Prime Number Theory Symposium

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- **Properties.** Weight 12, normalized, cuspidal newform for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. The Fourier coefficients $\tau(n)$ are **Ramanujan's tau function**.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $\tau(n)$ | 1 | $-24$ | 252 | $-1472$ | 4830 | $-6048$ | $\cdots$ |

# Modular forms

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- This is unsolved, but what can we say about the proportion of integers $n \geq 1$ such that $\tau(n) \neq 0$? Let

$$D_\Delta := \lim_{x \to \infty} \frac{\#\{n \leq x : \tau(n) \neq 0\}}{x}.$$

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- Serre (1981) shows that
$$D_\Delta = \prod_{\substack{p \text{ prime} \\ \tau(p)=0}} \left(1 - \frac{1}{p+1}\right)$$
$$= \prod_{\substack{p \le X_0 \\ \tau(p)=0}} \left(1 - \frac{1}{p+1}\right) \exp\left(-\int_{X_0}^{\infty} \frac{\pi_\Delta(x)}{x(x+1)} dx\right).$$

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- In order to bound $D_\Delta$ below, it suffices to:
  - Choose $X_0$ sufficiently large.
  - Bound $\pi_\Delta(x)$ above, $x \geq X_0$.

# Bounding $D_\Delta$

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- **Very large ($x > X_0$). We need an explicit upper bound for $\pi_\Delta(x)$ in this range.**

# Our goal

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- **Idea.** Relate $\pi_\Delta(x)$ to algebraic number theory using connections between modular forms and Galois representations.
- We make explicit an effective Lang-Trotter bound proved in work of Thorner-Zaman (2018).

# Galois representations

- Let $\ell \neq p$ be a prime. By Deligne (1969), there exists a representation

$$\rho_{\Delta,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\ell)$$

such that $\rho_{\Delta,\ell}(\mathrm{Frob}_p)$ has characteristic polynomial

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(By Serre and Swinnerton-Dyer (1972), $\rho_{\Delta,\ell}$ is surjective for $\ell > 691$.)

# Reduction to a Chebotarev problem

- For $\ell$ a prime, define

$$\pi_\Delta(x,\ell) := \#\left\{p \le x \;\middle|\; \tau(p) \equiv 0 \pmod{\ell},\ \left(\frac{\tau(p)^2 - 4p^{11}}{\ell}\right) = 1\right\}.$$

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- By a sieving argument of Wan (1990), we can choose primes $\ell_1, \ldots, \ell_t$ such that

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i.e. $\mathrm{Frob}_p \in C$, so

$$\pi_\Delta(x,\ell) = \#\{p \le x \mid \mathrm{Frob}_p \in C\}$$

where $C \subset \mathrm{Gal}(L_\ell/\mathbb{Q})$ is the union of conjugacy classes of traceless matrices with distinct eigenvalues in $\mathbb{F}_\ell$. **This is a Chebotarev problem!**

# The Chebotarev density theorem

- Let $L/K$ a Galois extension of number fields, $C \subset \mathrm{Gal}(L/K)$ a conjugacy class. Let

$$\pi_C(x, L/K) := \#\left\{\mathfrak{p} \subset \mathcal{O}_K \text{ prime} \mid \mathrm{N}\mathfrak{p} \leq x,\ \mathrm{Frob}_{\mathfrak{p}} = C\right\}.$$

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- **Our Chebotarev variant of the Brun-Titchmarsh theorem.** If $x \gg_{L/K} X_1$, then

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**A small range $X_1$ is crucial for applications.**

# Bounding $\pi_C(x, L/K)$

- Work of Murty-Murty-Saradha (1988) allows comparison of $\pi_C(x, L/K)$ with $\pi_{C'}(x, L/F)$ for an intermediate extension $F$ with abelian $L/F$.

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- For our $GL_2(\mathbb{F}_\ell)$ extension, work of Zywina (2015) allows comparison of $\pi_C(x, L/\mathbb{Q})$ with $\pi_{C'}(x, L^H/L^B)$ for $B$ the Borel (upper triangular) and $H$ the subspace of matrices with equal eigenvalues. $B/H$ is abelian.

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- **Class field theory.** Assume $L/K$ is abelian. Let $\mathfrak{f} = \mathfrak{f}_{L/K}$ be the Artin conductor of $L/K$ and $I_\mathfrak{f}/P_\mathfrak{f}$ the ray class group mod $\mathfrak{f}$. By Artin reciprocity $I_\mathfrak{f}/P_\mathfrak{f} \simeq \mathrm{Gal}(L/K)$ sending $\mathfrak{p}$ to $\mathrm{Frob}_\mathfrak{p}$, so it suffices to bound

$$\pi_C(x, L/K) = \#\{\text{prime ideal } \mathfrak{p} \in \text{coset of ray class group} \mid \mathrm{N}\mathfrak{p} \leq x\}.$$

# Bounding $\pi_C(x, L/K)$

- **Selberg sieve.** Compares a prime ideal counting problem to an integral ideal counting problem via inclusion-exclusion. It then suffices to fix an integral ideal $\mathfrak{d}$ of $K$ and bound

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- **Character orthogonality.** We can filter out ideals in a given coset:

$$\mathbb{1}_{\text{coset}} = \frac{1}{[L:K]} \sum_{\chi \in \widehat{\mathrm{Gal}(L/K)}} \bar{\chi}(\text{coset})\chi.$$

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Summing over ideals yields the count

$$\sum_{\substack{\mathrm{N}\mathfrak{n} \leq x \\ \mathfrak{d} | \mathfrak{n}}} \mathbb{1}_{\text{coset}}(\mathfrak{n}) = \frac{1}{[L:K]} \sum_{\chi \in \widehat{\mathrm{Gal}(L/K)}} \bar{\chi}(\text{coset}) \sum_{\substack{\mathrm{N}\mathfrak{n} \leq x \\ \mathfrak{d} | \mathfrak{n}}} \chi(\mathfrak{n}).$$

# Bounding $\pi_C(x, L/K)$

- Multiply by a **smooth** test function (Thorner–Zaman), and apply **Mellin inversion** to the Hecke $L$-function $L(s, \chi)$.

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- **Shift the contour and bound Hecke $L$-functions** in the critical strip by the Phragmén–Lindelöf principle.

# Our results

Choose $X_0$, apply Chebotarev bounds to the extensions $L_{\ell_i}/\mathbb{Q}$ arising from the $\ell_1, \ldots, \ell_t$-adic representations to obtain the following.

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**Theorem (Hu–I–Shashkov, 2023)**

If $x \geq e^{e^{16}}$, then

$$\pi_\Delta(x) \leq (3.01 \times 10^{-10}) \frac{x(\log \log x)^2}{(\log x)^2}.$$

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**Theorem (Hu–I–Shashkov, 2023)**

**We have $\tau(n) \neq 0$ for 99.99999999985% of positive integers.**

This is the first known positive unconditional lower bound for $D_\Delta$.

# Acknowledgements

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