Mean values of Hardy's Z-function and weak Gram's laws

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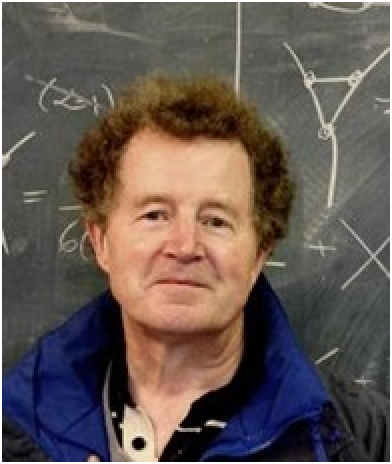
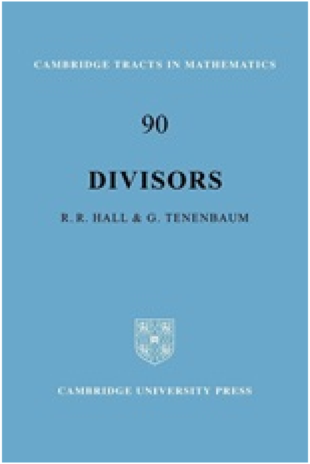
Joint work with Richard R. Hall (York)

Richard R. Hall





Outlines

- Hardy's Z-function
- Gram's points & Gram's Law
- Mean values of $Z(t)$ at Gram's points
- Weak Gram's Laws

Hardy's Z-function

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- Define $\chi(\frac{1}{2}+it)=e^{-2i\theta(t)}$. Hardy's Z-function:

$$Z(t):=e^{i\theta(t)}\zeta(\tfrac{1}{2}+it).$$

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$$Z(t) = 0 \quad \longrightarrow \quad \zeta(\tfrac{1}{2} + it) = 0.$$

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Gram's observations:

1. $\Re\zeta(\frac{1}{2} + it)$ is "rarely" negative.
2. $\Im\zeta(\frac{1}{2} + it)$ oscillates regularly between positive and negative values.
3. $\zeta(\frac{1}{2} + ig_n) > 0$, i.e. $Z(g_n)$ changes sign, for all $n \leq 15$.
4. There is exactly 1 zero of $\zeta(\frac{1}{2} + it)$ in (g_n, g_{n+1}) for $n \leq 15$.

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- $k = 0$: fourth moment of $\zeta(s)$.

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- Partial summation $\longrightarrow$ Theorem 1.

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- Gram's Law fails a positive proportion of times.
- It is not known if Gram's Law holds infinitely often.

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Theorem 3:

The Weak Gram's Law holds a positive proportion of times.

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- Let

$$\mathcal{S}_1 := - \sum_{T \le g_\nu \le 2T} Z(g_\nu) Z(g_{\nu+1}) \Big| M(\tfrac{1}{2} + i g_\nu) M(\tfrac{1}{2} + i g_{\nu+1}) \Big|^2$$

and

$$\mathcal{S}_2 := \sum_{T \le g_\nu \le 2T} Z(g_\nu)^4 \Big| M(\tfrac{1}{2} + i g_\nu) \Big|^8.$$

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- Holder's inequality:

$$
\begin{aligned}
&- \sum_{T\le g_\nu\le 2T} Z(g_\nu)Z(g_{\nu+1})\Big|M(\tfrac12+ig_\nu)M(\tfrac12+ig_{\nu+1})\Big|^2\\
&\le - \sum_{\substack{T\le g_\nu\le 2T\\ Z(g_\nu)Z(g_{\nu+1})<0}} Z(g_\nu)Z(g_{\nu+1})\Big|M(\tfrac12+ig_\nu)M(\tfrac12+ig_{\nu+1})\Big|^2\\
&\le \Big(\#\{T\le g_\nu\le 2T : Z(g_\nu)Z(g_{\nu+1})<0\}\Big)^{1/2}\\
&\qquad \times \bigg(\sum_{T\le g_\nu\le 2T} Z(g_\nu)^4\Big|M(\tfrac12+ig_\nu)\Big|^8\bigg)^{1/4}\\
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\end{aligned}
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- $k = 1$: more complicated.
- The lower bound for $\mathcal{S}_1$ is much more difficult,

$$\mathcal{S}_1 = - \sum_{T \le g_\nu \le 2T} Z(g_\nu) Z(g_{\nu+1}) \Big| M(\tfrac{1}{2}+ig_\nu) M(\tfrac{1}{2}+ig_{\nu+1}) \Big|^2.$$

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Theorem 4:

$$meas\{t \in [0, T] : X(t) > 0\} \gg T,$$
$$meas\{t \in [0, T] : X(t) < 0\} \gg T.$$

- Similar idea to Selberg (1942), and Gonek & Ivić (2017).