Mean values of long Dirichlet polynomials

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General mean values

- We are interested in mean values of Dirichlet polynomials

$$\sum_{n \leqslant N} a(n) n^{-it}.$$

Primarily, since they approximate L-functions in regions where their Dirichlet series do not converge.

- Typically, the length of the sum N will depend on the height t. For instance, given large $t \in [T, 2T]$

$$\zeta(\tfrac{1}{2} + it)^k \approx \sum_{n \leqslant T^{k/2}} d_k(n) n^{-1/2-it},$$

$$\log \zeta(\tfrac{1}{2} + it) \approx \sum_{p \leqslant T^{\epsilon}} p^{-1/2-it}.$$

- We consider the mean value

$$\int_T^{2T} \Big| \sum_{n \leqslant N} a(n) n^{-it} \Big|^2 dt$$

for large T, general coefficients $a(n) \ll n^{\epsilon}$ and N dependent on T.

- Montgomery–Vaughan mean value theorem:

$$\int_T^{2T} \Big| \sum_{n \leqslant N} a(n) n^{-it} \Big|^2 dt = (T + O(N)) \sum_{n \leqslant N} |a(n)|^2.$$

So if $N = o(T)$ we get an asymptotic formula. In a lot of cases, actually expect $\sim cT \sum |a(n)|^2$, even for N much larger than T.

- Terminology: If

$N \ll T$	“Short Dirichlet polynomial"
$N \gg T$	“Long Dirichlet polynomial".

- To see what's needed for long polys let's compute:

$$\begin{aligned}\int_T^{2T}\Big|\sum_{n\leqslant N} a(n)n^{-it}\Big|^2 dt &= \sum_{m,n\leqslant N} a(m)\overline{a(n)}\int_T^{2T}(m/n)^{-it}dt \\ &= T\sum_{n\leqslant N}|a(n)|^2 + \sum_{\substack{m\neq n\\ m,n\leqslant N}} a(m)\overline{a(n)}\frac{(m/n)^{-it}}{-i\log(m/n)}\Big|_T^{2T}\end{aligned}$$

- Worst case $m = n+1$ and $n \approx N$ then

$$\log(m/n) = \log(1+\frac{1}{n}) \approx \frac{1}{N}.$$

Can give a large contribution, a lot of care is needed.

- To understand long Dirichlet polynomials we need precision in off-diagonals $m-n=h\neq 0$. In turn requires understanding the correlation sums
$$\sum_{n\leqslant x} a(n)\overline{a(n+h)}.$$
Typically need full asymptotic formula with power saving in error term.

- For example, when $a(n)=d_k(n)$ and $k=2$ these are reasonably well understood:
$$\sum_{n\leqslant x} d(n)d(n+h)=c_2(h)x(\log x)^2+c_1(h)x\log x+c_0(h)x+O_h(x^{\theta+\epsilon}).$$
Ingham, Estermann, Heath-Brown ($\theta=5/6$ uniformly in $h\ll x^{5/6}$ via Weil's bound), then using spectral theory for sums of Kloosterman sums: Deshouillers–Iwaniec ($\theta=2/3$ and fixed $h\geqslant 1$), Kusnetsov, Motohashi ($\theta=2/3$ with uniformity in h).

- Divisor functions with $k \geqslant 3$ are more mysterious. It is conjectured that

$$\sum_{n \leqslant x} d_k(n) d_k(n+h) = x\, P_{k,h}(\log x) + O_{k,h}(x^{\theta})$$

for some polynomial $P_{k,h}(y)$ of degree $2(k-1)$ and $\theta < 1$.

- For sixth moment of the Riemann zeta function would be applying asymptotic with $x = T^{3/2}$ and $|h| \leqslant T^{1/2}$. Error term in divisor problem translates, pretty much, directly into error term for moment problem. So in order for the error term of the sixth moment to be $\ll T$, we would require

$$\theta \leqslant 2/3$$

uniformly in such h.

- For eighth moment would have $x = T^2$ so would require squareroot cancellation $\theta = 1/2$. For higher moments one requires cancellation *amongst* the error terms when averaged over h.

- When $a(n)$ is indicator function of the primes, or rather $a(n) = \Lambda(n)$, this is part of the famous Hardy–Littlewood k-tuples conjecture. In a strong form this predicts

$$\sum_{n \leqslant x} \Lambda(n)\Lambda(n+h) = \mathfrak{S}(h)x + O_h(x^{1/2+\epsilon})$$

  for some constant $\mathfrak{S}(h)$.

- We examine two applications/occurrences of this conjecture in zeta function theory.

Prime Dirichlet polynomials I: Pair correlation of zeros

- Want to understand interactions between nearby zeros at some large height T. So consider

$$\sum_{0\leqslant\gamma,\gamma'\leqslant T} f\Big(\frac{\gamma-\gamma'}{2\pi/\log T}\Big)$$

for some f concentrated around 0.

- Unfolding the inverse Fourier transform this is

$$\int_{-\infty}^{\infty}\widehat{f}(\alpha)\sum_{0\leqslant\gamma,\gamma'\leqslant T} T^{i\alpha(\gamma-\gamma')}d\alpha.$$

Inner sums are now separable, but a little too erratic. Also, applying explicit formula will give pointwise problem on Dirichlet polynomials - not much better. Idea: Can smooth sum and make it a mean value at the same time by introducing some averaging.

- Since f is concentrated can introduce extra (less concentrated) factor at little cost:
$$\sum_{0\leqslant\gamma,\gamma'\leqslant T} f\Big(\frac{\gamma-\gamma'}{2\pi/\log T}\Big)w(\gamma-\gamma')$$
where $w(u)=4/(4+u^2)$.
- Then inner sum is
$$\sum_{0\leqslant\gamma,\gamma'\leqslant T} T^{i\alpha(\gamma-\gamma')}w(\gamma-\gamma')=\frac{2}{\pi}\int_0^T\bigg|\sum_{\gamma}\frac{T^{i\alpha\gamma}}{1+(\gamma-t)^2}\bigg|^2dt+\cdots$$
since $w(u)=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+t^2}\frac{1}{1+(u-t)^2}dt$. So we have a smoothed, averaged sum now on the right.

- Apply explicit formula: for $y \geqslant 1$,

$$\sum_{\gamma} \frac{y^{i\gamma}}{1+(\gamma-t)^2} \approx -y^{-1/2+it} \sum_{y\leqslant n\leqslant 2y} \Lambda(n)n^{-it} + \frac{y^{1/2}}{1+t^2} + \frac{\log(2+|t|)}{y^{1-it}}$$

- With $y = T^{\alpha}$, applying MV the mean square of first term is

$$\begin{aligned}\frac{1}{T^{\alpha}}\int_0^T \Big|\sum_{n\asymp T^{\alpha}} \Lambda(n)n^{-it}\Big|^2 dt =& \frac{1}{T^{\alpha}}(T+O(T^{\alpha})) \sum_{n\asymp T^{\alpha}} \Lambda(n)^2 \\ \sim& \alpha T \log T\end{aligned}$$

provided $0 < \alpha < 1$. Note we are only using short sum information here.

- In this way, Montgomery showed that the normalised function

$$F(\alpha) := \frac{2\pi}{T\log T}\sum_{0\leqslant\gamma,\gamma'\leqslant T} T^{i\alpha(\gamma-\gamma')}w(\gamma-\gamma')$$

  satisfies

$$F(\alpha)\sim\alpha+T^{-2\alpha}\log T$$

  for $0<\alpha<1$. Also, more trivially F is even so we understand $F(\alpha)$ for $-1<\alpha<1$.

- Returning to the original sum:

$$\sum_{0\leqslant\gamma,\gamma'\leqslant T} f\Big(\frac{\gamma-\gamma'}{2\pi/\log T}\Big)=\frac{T}{2\pi}\log T\int_{-\infty}^{\infty}\widehat{f}(\alpha)F(\alpha)d\alpha+\cdots$$

  So can understand the right hand side provided $\mathrm{supp}\,\widehat{f}\subset[-1,1]$.

- To understand local properties of zeros we would like to take f as concentrated as possible. But then $\widehat{f}$ gets wider so it is desirable to understand $F(\alpha)$ outside the range $[-1,1]$.

- For this we need to understand mean values of longer Dirichlet polynomials. Assuming the Hardy–Littlewood conjecture

$$\sum_{n\leqslant x}\Lambda(n)\Lambda(n+h)=\mathfrak{S}(h)x+O_h(x^{1/2+\epsilon})$$

  Montgomery showed that

$$F(\alpha)\sim 1$$

  for $1\leqslant\alpha\leqslant 2-\epsilon$ and conjectured that this remains true for larger α.

- For larger α the error terms in the HL conjecture would probably have to exhibit cancellation amongst themselves when averaged over h, similarly to the higher moments of the zeta function.

- In any case, Montgomery's conjecture implies all sorts of nice consequences for zeros.

- Pair correlation function and connection with random matrix theory:

$$\frac{2\pi}{T\log T}\sum_{\frac{2\pi a}{\log T}\leqslant\gamma-\gamma'\leqslant\frac{2\pi b}{\log T}}1\sim\int_a^b\Big(1-\Big(\frac{\sin\pi u}{\pi u}\Big)^2\Big)du$$

- There exist arbitrarily small normalised gaps between zeros:

$$\liminf_{\gamma\to\infty}\frac{\gamma^+-\gamma}{2\pi/\log\gamma}=0$$

  In a quantitative form this implies no Siegel zeros.

- Almost all zeros are simple.

Prime Dirichlet polynomials II: Value distribution of $\log\zeta(\frac{1}{2}+it)$

- Selberg's Central Limit Theorem: for fixed $V \in \mathbb{R}$,

$$\Phi(V) := \frac{1}{T}\mu\left(t \in [T, 2T] : \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} \geqslant V\right) \sim \frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-x^2/2}dx$$

  as $T \to \infty$ where μ denotes Lebesgue measure. So the zeta function is as large as $\exp(c\sqrt{\log\log T})$ and as small as $\exp(-c\sqrt{\log\log T})$ a positive proportion of the time.

- For the $2k$th moment $\int_T^{2T}|\zeta(\frac{1}{2}+it)|^{2k}dt$ it is values of size

$$(\log T)^k = \exp(k\log\log T)$$

  that dominate, so of interest to understand CLT for large V ($\approx k\sqrt{\log\log T}$).

- Soundararajan (2009) gave large deviation bounds on the Riemann hypothesis:

$$\Phi(V) \ll \exp(-(1-\epsilon)V^2/2)$$

for $V \ll \sqrt{\log\log T}\log_3 T$. This allowed for near sharp bounds on the $2k$th moment of the zeta function:

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k} dt \ll T(\log T)^{k^2+\epsilon}.$$

- It is of interest to push these large deviations are far as possible since the global maximum of the zeta function is unknown. It is conjectured to be around $\exp(\sqrt{\log T \log\log T})$, which gives a huge range for which we don't know the distribution:

$$\sqrt{\log\log T}\log_3 T \ll V \ll \sqrt{\log T}$$

- Main tool for central limit theorem: The explicit formula

$$\log\zeta(\tfrac{1}{2}+it)=\sum_{p\leqslant X}\frac{1}{p^{1/2+it}}-\sum_{\rho}\int_{1/2}^{\infty}\frac{X^{\rho-\sigma-it}}{\rho-\sigma-it}d\sigma+\cdots$$

  for large X.
- In the sum over zeros $X^{\rho-\sigma-it}\ll X^{1/2-\sigma}\ll 1$ on RH, and singularities are logarithmic so can be integrated dt.
- Sum is localised over zeros ρ with imaginary part within distance $1/\log X$ of t. This gives around $\log t/\log X$ terms on average. So on average we have

$$\log\zeta(\tfrac{1}{2}+it)=\sum_{p\leqslant X}\frac{1}{p^{1/2+it}}+O\Big(\frac{\log t}{\log X}\Big).$$

- Note we can choose $X=T^{\epsilon}$ to give a short sum plus a bounded error.

- In particular, for $2k$th moment choose $X = T^{1/k}$ so that sum over primes is short. Then

$$\frac{1}{T}\int_T^{2T} |\log \zeta(\tfrac{1}{2}+it)|^{2k} dt = \frac{1}{T}\int_T^{2T} \Big|\sum_{p\leqslant X} \frac{1}{p^{1/2+it}}\Big|^{2k} dt + \cdots$$
$$\sim \sum_{\substack{p_1\cdots p_k = p_{k+1}\cdots p_{2k} \\ p_j \leqslant X}} \frac{1}{(p_1\cdots p_{2k})^{1/2}}$$
$$= k!\Big(\sum_{p\leqslant X}\frac{1}{p}\Big)^k + \cdots$$
$$= k!(\log\log T)^k + \cdots$$

  i.e. (Complex) Gaussian.
- So p^{-it} look like independent random variables on unit circle.
- Shortness of polynomials limits large deviation results. Would be nice to be able to compute long polynomials but don't have access to the correlation sums/Hardy–Littlewood k-tuples conjecture.

Mean values of longer Dirichlet polynomials

- In special cases can compute mean values of long Dirichlet polynomials over primes assuming RH. Makes use of weights. Need
  - $\phi_X : \mathbb{R} \to \mathbb{R}$ bounded and of compact support in $[0, X]$.
  - The Mellin transform $\widehat{\phi}_X(s) = \int_0^\infty \phi_X(u) u^{s-1} du$ satisfies
    $$\widehat{\phi}_X(s) \ll \frac{X^{\Re(s)}}{|s|^2}$$
    for fixed $\Re(s)$ as $\Im(s) \to \infty$.
  - For fixed $\Re(s) > 0$, we have
    $$\frac{1}{\log X} \widehat{\phi}_X(s/\log X) \ll \frac{1}{1 + \Im(s)^2}.$$
- Satisfied by the pair
  $$\phi(u) = \mathbb{1}_{0<u\leqslant X} \cdot \left(1 - \tfrac{\log u}{\log X}\right), \qquad \widehat{\phi}_X(s) = X^s/(s^2 \log X).$$
  But unfortunately not by $\phi_X(u) = \mathbb{1}_{0<u\leqslant X}$ and its transform $\widehat{\phi}_X(s) = X^s/s$.

Theorem

Assume the Riemann hypothesis and let ϕ_X be as above. Then for $X \leqslant T^4$ and $k \in \mathbb{N}$ we have

$$\frac{1}{T}\int_T^{2T}\bigg|\sum_{p\leqslant X}\frac{\phi_X(p)}{p^{1/2+it}}\bigg|^{2k}dt = k!\bigg(\sum_{p\leqslant X}\frac{\phi_X(p)^2}{p}\bigg)^k + O((ck)^{5k}(\log\log T)^{k-1/2})$$

and

$$\frac{1}{T}\int_T^{2T}\bigg(\mathfrak{F}\sum_{p\leqslant X}\frac{\phi_X(p)}{p^{1/2+it}}\bigg)^{2k}dt = c_k\bigg(\frac{1}{2}\sum_{p\leqslant X}\frac{\phi_X(p)^2}{p}\bigg)^k + O((ck)^{5k}(\log\log T)^{k-1/2})$$

where $\mathfrak{F}$ denotes either the real or imaginary part and

$$c_k = \begin{cases}\frac{(2k)!}{2^k k!} & 2k \text{ is even,}\\ 0 & 2k \text{ is odd.}\end{cases}$$

- Rough idea: Recall that on average we have

$$\log\zeta(\tfrac{1}{2}+it)=\sum_{p\leqslant X}\frac{1}{p^{1/2+it}}+O\Big(\frac{\log t}{\log X}\Big).$$

  One can check the zero sum is not very sensitive to long or short X - so this should hold regardless of whether $X\gg T$ or not.

- So take short $Y=T^{1/k}$ and compare the above at X and Y to give

$$\sum_{p\leqslant X}\frac{1}{p^{1/2+it}}=\sum_{p\leqslant Y}\frac{1}{p^{1/2+it}}+O\Big(\frac{\log t}{\log Y}\Big)$$

  on average i.e. we can replace a long poly with a short one on average.

- In practice is best to avoid zeros entirely, so we use a contour integral and shift very close to the half-line, but not past it.

- Proof sketch: Work with the weight

$$\phi_X(u) = \mathbb{1}_{0<u\leqslant X} \cdot \left(1 - \tfrac{\log u}{\log X}\right), \qquad \widehat{\phi}_X(s) = X^s/(s^2 \log X).$$

  Can accept errors of $O(1)$ throughout.
- By Mellin inversion and RH

$$\begin{aligned}\sum_{p\leqslant X} \frac{\phi_X(p)}{p^{1/2+it}} =& \frac{1}{2\pi i \log X} \int_{1-i\infty}^{1+i\infty} \log\zeta(\tfrac{1}{2}+it+s) X^s \frac{ds}{s^2} \\ =& \frac{1}{2\pi i \log X} \int_{1/\log X - i\infty}^{1/\log X + i\infty} \log\zeta(\tfrac{1}{2}+it+s) X^s \frac{ds}{s^2} + O\Big(\frac{X^{1/2}}{t^2}\Big).\end{aligned}$$

- So for $X \leqslant T^4$ the contribution from the pole can be ignored since $t \in [T, 2T]$. Note smoothings with more transform decay can allow for larger X.
- Then truncate integral at height $\Im(s) = \pm 1$. Logarithm of zeta is $\ll \log t$ and we have factor of $1/\log X$ out front, so negligible contribution.

- Then after substitution and cleaning up

$$\sum_{p\leqslant X}\frac{\phi_X(p)}{p^{1/2+it}}\sim\frac{1}{2\pi}\int_{-\log X}^{\log X}\log\zeta(\sigma_0+it+\tfrac{iy}{\log X})e^{1+iy}\frac{dy}{(1+iy)^2}.$$

with $\sigma_0=\frac{1}{2}+\frac{1}{\log X}$.

- Raising to absolute $2k$th power and integrating with respect to t we need to compute correlations

$$\int_T^{2T}\prod_{j=1}^{k}\log\zeta(\sigma_0+it+i\tilde{y}_j)\prod_{j=k+1}^{2k}\log\zeta(\sigma_0-it-i\tilde{y}_j)dt$$

for small shifts $\tilde{y}_j$ satisfying $|\tilde{y}_j|\leqslant 1$.

- Now input short sum mean approximation

$$\log\zeta(\sigma_0+it+i\tilde{y}_j)=\sum_{p\leqslant Y}\frac{1}{p^{\sigma_0+it+i\tilde{y}_j}}+O(1)$$

with $Y=T^{1/k}$, say.

- Short polynomials and so diagonals give major contribution. Correlation integral is given by

$$\sum_{\substack{p_1 \cdots p_k = p_{k+1} \cdots p_{2k} \\ p_j \leqslant Y}} \frac{\prod_{j=1}^{k} p_j^{-i\tilde{y}_j} \cdot \prod_{j=k+1}^{2k} p_j^{i\tilde{y}_j}}{(p_1 \cdots p_{2k})^{\sigma_0}}.$$

- Mellin integrals over y_j are separable so can compute to give result:

$$\frac{1}{2\pi} \int_{-\log X}^{\log X} p^{-1/\log X - iy/\log X} e^{1+iy} \frac{dy}{(1+iy)^2} \sim \phi_X(p)$$

so the sum is

$$\sim k! \Big(\sum_{p \leqslant Y} \frac{\phi_X(p)^2}{p} \Big)^k \qquad \square$$

- Smoothing very helpful here. With brutal cut-off $\mathbb{1}_{n\leqslant X}$ and its transform X^s/s we would have difficulty truncating. Also, mean $O(1)$ error term in prime sum approximation would cause blow ups:

$$\int_{-\log X}^{\log X} O(1)\cdot e^{1+iy}\frac{dy}{1+iy} \ll \log\log X.$$

Large deviations

- Unfortunately doesn't give new large deviations results for $\log \zeta$ - haven't input any new information! Nevertheless, can still get large deviations for the polynomials.
- The $O(1)$ error term in prime sum approximation leads to poor dependency on k in error terms:

  $$O((ck)^{5k}(\log\log T)^{k-1/2}).$$

  Not so useful for large deviations.
- Can use more efficient methods to get these. For the real part, we try to apply Soundararajan's upper bound

  $$\log|\zeta(\tfrac{1}{2}+it)| \leqslant \Re \sum_{p\leqslant Y} \frac{1}{p^{1/2+it}} + \frac{\log t}{\log Y}.$$

  more directly.

- First modify $\widehat{\phi}_X(s)$ so that $\widehat{\phi}_X(iy)$ is positive. E.g.

$$\mathbb{1}_{n\leqslant X}\cdot\left(1-\tfrac{\log n}{\log X}\right)=\frac{1}{2\pi i\log X}\int_{c-i\infty}^{c+i\infty}\left(\frac{X}{n}\right)^s\frac{ds}{s^2}$$

$$=\frac{1}{2\pi i\log X}\int_{c-i\infty}^{c+i\infty}\left[\left(\frac{X}{n}\right)^s-\frac{2}{n^s}+\frac{X^{-s}}{n^s}\right]\frac{ds}{s^2}$$

$$=\frac{1}{2\pi i\log X}\int_{c-i\infty}^{c+i\infty}\frac{1}{n^s}\sin^2\left(\tfrac{1}{2i}s\log X\right)\frac{ds}{s^2}.$$

- Then perform same argument as before shifting to $\Re(s)=0$:

$$\sum_{p\leqslant X}\frac{\phi_X(p)}{p^{1/2+it}}\sim\frac{1}{2\pi}\int_{-\log X}^{\log X}\log\zeta\left(\tfrac{1}{2}+it+\tfrac{iy}{\log X}\right)\frac{\sin^2(\frac{1}{2}y)}{y^2}dy.$$

- Take real parts, input upper bound and analyse the tails of each component individually following Soundararajan's argument fairly closely. Extra integral over y can be dealt with easily via Hölder's inequality.

Theorem

Assume the Riemann hypothesis. Let ϕ_X be as above and let $X \leqslant T^4$. Then

$$\frac{1}{T}\mu\left(t \in [T, 2T] : \frac{\Re \sum_{p \leqslant X} \phi_X(p) p^{-1/2-it}}{\sqrt{\frac{1}{2}\log\log T}} \geqslant V\right)$$
$$\ll \exp(-(1-\epsilon)V^2/2) + \exp(-CV\sqrt{\log\log T}\log V).$$

- In the range $V \ll \sqrt{\log_2 T}\log_3 T$ have Gaussian bounds - results of this quality were previously restricted to $X \leqslant T^{1/\sqrt{\log\log T}}$.
- In the full range of V have $e^{-cV\log V}$ - previously only available for $X \leqslant (\log T)^\theta$ with $\theta \leqslant 2$.
- Restricted to positive values of real part. Similar result holds for imaginary part but with no restriction on positive values.

Other polynomials

- Rough principal: Express *long* Dirichlet polynomial as contour integral in efficient way. Approximate $\log \zeta$ by *short* Dirichlet polynomial and then compute.
- Can apply this to other long polynomials. Consider

$$\sum_{p \leqslant X} \frac{\phi_X(p) \log p}{p^{1/2+it}}$$

  which approximates $\frac{\zeta'}{\zeta}(\frac{1}{2} + it)$.
- Look at mean square. Expect this to be

$$\approx \sum_{p \leqslant X} \frac{\log^2 p}{p} \asymp (\log T)^2.$$

Theorem

Assume the Riemann hypothesis. Suppose $X \leqslant T^4$ *and let* $\theta = \frac{\log X}{\log T}$ *. Then*

$$\frac{1}{T}\int_T^{2T}\Big|\sum_{p\leqslant X}\frac{\phi_X(p)\log p}{p^{1/2+it}}\Big|^2 dt \sim \sum_{p\leqslant \min(T,X)}\frac{\phi_X(p)^2(\log p)^2}{p}$$

$$+\mathbb{1}_{X\geqslant T}\cdot(\log T)^2\int_1^{\theta}F(\alpha)\phi_X(e^{\alpha/\theta})^2 d\alpha$$

- First term is diagonal contribution and second term represents off-diagonal contribution.
- Montgomery's conjecture $F(\alpha)\sim 1$ gives asymptotic $\sim c(\log T)^2$
- Without this we make use of average results: $\int_b^{b+1}F(\alpha)d\alpha \asymp 1$ to give the order $\asymp (\log T)^2$.

- Similar contour integral argument then integration by parts gives

$$\sum_{p\leqslant X}\frac{\phi_X(p)\log p}{p^{1/2+it}}\sim\frac{1}{2\pi i\log X}\int_{1/\log X-i\log X}^{1/\log X+i\log X}\log\zeta(\tfrac{1}{2}+it+s)\frac{d}{ds}\left[\frac{X^s}{s^2}\right]ds.$$

- Derivative adds extra factor of $\log X$ and we're looking for $\sim c(\log T)^2$. So $O(1)$ term in approximation $\log\zeta=\sum_p+O(1)$ would lead to $O((\log X)^2)$ - not sufficient.
- Need very sharp estimates for correlation integrals.
- Using extra contour integral argument can restrict to imaginary part of log and then only need to consider

$$\int_T^{2T}S(t+y_1)S(t+y_2)dt$$

where $S(t)=\frac{1}{\pi}\Im\log\zeta(\frac{1}{2}+it)$ for $|y_j|\ll\log T$.

- Very precise result of Goldston (1987) gives lower order terms in

$$\int_T^{2T} S(t)^2 dt.$$

- These lower order terms involve Montgomery's function $F(\alpha)$.
- Using Goldston's arguments for precise estimates of mean square of $S(t)$, we can get precise estimates for our correlation integral to give the result.

Thanks!