Birational geometry of quiver varieties

Gwyn Bellamy

jt. with A. Craw, T. Schedler (S. Rayan and H. Weiss)

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Plan

- Introduction
- Quiver varieties
- Anisotropic roots
- Isotropic roots
- Hyperpolygon spaces

Introduction

- Quiver varieties, as introduced by Nakajima, are ubiquitous in geometric representation theory.
- Large class of examples of symplectic singularities, together with an associated symplectic resolution given by variation of geometric invariant theory (VGIT).

Questions:

(A) Can one obtain all symplectic resolutions via VGIT?

(B) What is the birational transformation that occurs when we cross a GIT wall?

Quiver varieties

- $Q = (Q_0, Q_1)$ a finite quiver with double $\overline{Q}$.
- $v \in \mathbb{N}^{Q_0}$ dimension vector.
- Space of representations of dimension v:

$$\mathrm{Rep}(\overline{Q}, v) = \bigoplus_{a \in \overline{Q}_1} \mathrm{Hom}(\mathbb{C}^{t(a)}, \mathbb{C}^{h(a)}).$$

- Carries (Hamiltonian) action of $G(v) = \prod_{i \in Q_0} GL(\mathbb{C}^{v_i})$.
- Corresponding moment map

$$\mu \colon \mathrm{Rep}(\overline{Q}, v) \to \mathfrak{g}(v)$$

where $\mathfrak{g}(v) = \mathrm{Lie}\, G(v)$.

Quiver varieties

Definition

The quiver variety associated to (Q, v) is

$$\mathfrak{M}_0 := \mu^{-1}(0)/\!/G(v).$$

Proposition (B-Schedler)

$\mathfrak{M}_0$ has symplectic singularities.

Q. When does $\mathfrak{M}_0$ admit a symplectic resolution?

Factorization

- $\mathbb{Z}^{Q_0}$ has symmetric form $(-,-)$.
- Applying Crawley-Boevey's factorization,

$$\mathfrak{M}_0(v) \cong \mathfrak{M}_0(v_1) \times \cdots \times \mathfrak{M}_0(v_k)$$

  where each $v_i \leq v$ is either
  (1) $v_i = n\delta_i$ for δ_i minimal imaginary, $(\delta_i, \delta_i) = 0$; or
  (2) anisotropic root: $(v_i, v_i) < 0$.

- $\mathfrak{M}_0(v)$ admits a symplectic resolution iff every factor $\mathfrak{M}_0(v_i)$ admits a symplectic resolution.
- Hilbert schemes give symplectic resolutions in case (1).

Anisotropic roots

In the case where v is an anisotropic root, $(v, v) < 0$, have:

Theorem (B-Schedler)

$\mathfrak{M}_0(v)$ admits a symplectic resolution iff v indivisible or "$(2,2)$ case".

The "$(2,2)$ case" is $v = 2u$ with u indivisible, $(u, u) = -2$. This situation is exceptional.

Anisotropic roots - birational geometry

Assume v anisotropic and indivisible. Set

$$\Lambda = \left\{ \theta \in \mathbb{Q}^{Q_0} \,|\, \theta(v) = 0 \right\}.$$

For each $\theta \in \Lambda$, consider space

$$\mu^{-1}(0)^{\theta} = \{ M \in \mu^{-1}(0) \,|\, \theta(\dim M') \leq 0, \ \forall\, M' \text{ subrep } M \}$$

space of θ-semistable objects.

Definition

$$\mathfrak{M}_{\theta} := \mu^{-1}(0)^{\theta} /\!/ G(v).$$

Always a Poisson morphism $\mathfrak{M}_{\theta} \to \mathfrak{M}_0$.

Anisotropic roots - birational geometry

Λ^{reg} set of all $\theta \in \Lambda$ with $\mathfrak{M}_\theta$ smooth.

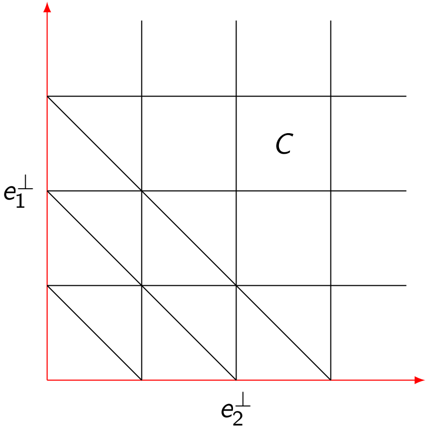
Proposition (B-Craw-Schedler = BCS)

Λ^{reg} complement to finitely many hyperplanes H_α.

- Hyperplanes H_α are either "interior" or "boundary", depending on α.
- Fix $C \subset \Lambda^{\text{reg}}$ a chamber and $\theta \in C$.
- C lies in a unique (closed) chamber F of the boundary arrangement.

Anisotropic roots - birational geometry

Slice to arrangement in $\Lambda = \mathbb{Q}^3$, showing chambers in F (boundary,interior).



Anisotropic roots - birational geometry

- Define $L_C : \Lambda \to \mathrm{Pic}(\mathfrak{M}_\theta)_{\mathbb{Q}}$ by

$$L_C(\vartheta) = \bigotimes_{i \in Q_0} (\det \mathcal{R}_i)^{\otimes \vartheta_i}$$

- Here $\mathcal{R}_i$ tautological bundle of rank v_i.

Theorem (BCS)

1. L_C is an isomorphism with $L_C(C) = \mathrm{Amp}(\mathfrak{M}_\theta)$.
2. $L_C = L_{C'}$ if $C, C' \subset F$.
3. $L_C(F) = \mathrm{Mov}(\mathfrak{M}_\theta)$.

Surjectivity of L_C requires McGerty-Nevins theorem on surjectivity of the Kirwan map.

Application

Corollary (BCS)

Let v be arbitrary. Every (projective) symplectic resolution of $\mathfrak{M}_0(v)$ is given by a quiver variety.

Need to exclude $(2,2)$ case above.

Corollary (BCS)

Assume v a root. If $\mathfrak{M}_0(v)$ admits a symplectic resolution then

$$\#\text{resolutions} = |\pi_0(F \cap \Lambda^{\text{reg}})|.$$

$\mathbb{Q}$-factorial terminalizations

Assume v is not indivisible.

Proposition (B-Schedler)

For generic $\theta \in \Lambda$, $\mathfrak{M}_\theta \to \mathfrak{M}_0$ is a $\mathbb{Q}$-factorial terminalization.

BCS:

- Chamber structure still exists.
- L_C is always injective.
- Know to be surjective in certain cases.
- Expect it always to be an isomorphism.

All results make sense in this generality provided L_C is an isomorphism.

Isotropic roots

- Q affine Dynkin quiver.
- $v = n\delta$ with δ minimal imaginary root.
- $\Delta = \{e_1, \dots, e_r\}$ simple roots in **finite** root system Φ.

Hyperplanes

- $\mathcal{A}_I = \{\beta + m\delta \,|\, \beta \in \Phi, \; -n < m < n, m \neq 0\}$.
- $\mathcal{A}_B = \{\delta\} \cup \Phi^+$.

Then

- H_α for $\alpha \in \mathcal{A}_I$ are "interior" hyperplane.
- H_α for $\alpha \in \mathcal{A}_B$ are "boundary" hyperplane.

Isotropic roots

Theorem (B-Craw)

- $\Lambda^{\mathrm{reg}} = \Lambda \smallsetminus \bigcup_\alpha H_\alpha$, where $\alpha \in \mathcal{A}_I \cup \mathcal{A}_B$.
- $F = \{\theta \in \Lambda \mid \theta(\delta) \geq 0, \theta(e_i) \geq 0, i = 1, \dots, r\}$.

W_Φ be the (finite) Weyl group of Φ.

Theorem (B-Craw)

- $W = \mathfrak{S}_2 \times W_\Phi$ acts on Λ with fundamental domain F.
- $\mathfrak{M}_C \cong \mathfrak{M}_{C'}$ iff $C' = w(C)$ some $w \in W$.

Application

- $\Gamma \subset SL(2,\mathbb{C})$ finite group associated to Q.
- $\mathfrak{S}_n \wr \Gamma = \mathfrak{S}_n \rtimes \Gamma^n$ acts on $\mathbb{C}^{2n}$.
- Symplectic resolution of quotient given by

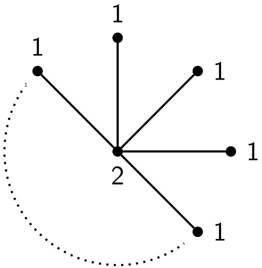
$$\mathrm{Hilb}^n\left(\widetilde{\mathbb{C}^2/\Gamma}\right) \to \mathbb{C}^{2n}/(\mathfrak{S}_n \wr \Gamma)$$

where $\widetilde{\mathbb{C}^2/\Gamma}$ minimal resolution of $\mathbb{C}^2/\Gamma$.

Corollary (B-Craw)

Every (projective) symplectic resolution of $\mathbb{C}^{2n}/(\mathfrak{S}_n \wr \Gamma)$ is of the form $\mathfrak{M}_\theta$ for some $\theta \in \Lambda^{\mathrm{reg}}$.

Hyperpolygon spaces

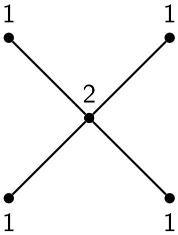


Let $n \geq 4$ and (Q, v) star quiver with n outer vertices.

- $\mathfrak{M}_\theta(n)$ a "hyperpolygon space".
- As a hyperhähler manifold, compactification of cotangent bundle of polygon moduli space.
- $\dim \mathfrak{M}_\theta = 2(n-3)$.

A quotient singularity

- Notice for $n = 4$, $\mathfrak{M}_0 \cong \mathbb{C}^2/\mathrm{BD}_8$.



A quotient singularity

- Notice for $n = 4$, $\mathfrak{M}_0 \cong \mathbb{C}^2/\mathrm{BD}_8$.

Theorem (B-Schedler)

The group $Q_8 \times_{\mathbb{Z}_2} D_8$ acts on $\mathbb{C}^4$ such that $\mathbb{C}^4/(Q_8 \times_{\mathbb{Z}_2} D_8)$ admits a symplectic resolution.

Theorem (B,Donten–Bury-Wiśniewski)

The quotient $\mathbb{C}^4/(Q_8 \times_{\mathbb{Z}_2} D_8)$ admits 81 (projective) symplectic resolutions.

A quotient singularity

Theorem (B-Craw-Rayan-Schedler-Weiss)

As symplectic singularities,

$$\mathbb{C}^4/(Q_8 \times_{\mathbb{Z}_2} D_8) \cong \mathfrak{M}_0(5).$$

Easy to recover count of 81 from hyperplane arrangement in Λ.

Theorem (B-Craw-Rayan-Schedler-Weiss)

For $n \geq 4$, we have $\Lambda \cong \mathbb{Q}^n$ with

- $\Lambda^{\text{reg}} = \{\theta \,|\, \theta_1 \pm \theta_2 \pm \cdots \pm \theta_n \neq 0,\ \theta_1, \ldots, \theta_n \neq 0\}$.
- $F = \{\theta \,|\, \theta_i \geq 0\}$.
- $W = \mathfrak{S}_2^n$.

The End!

Thanks for listening.