Exact Lagrangians in conical symplectic resolutions

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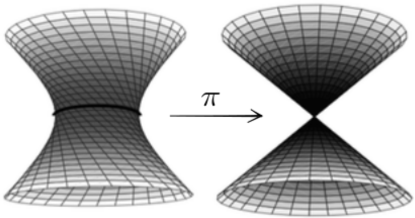
Overview

A first example $\pi : T^*\mathbb{C}P^1 \to \mathbb{C}^2/(\mathbb{Z}/2)$

- $\pi : T^*\mathbb{C}P^1, \to \mathbb{C}^2/(\mathbb{Z}/2)$ blow up
- $(-1) \curvearrowright (z_1, z_2) = (-z_1, -z_2)$
- $t \cdot (z_1, z_2) = (tz_1, tz_2)$ contracts $\mathbb{C}^2/(\mathbb{Z}/2)$ to a point.
- Action lifts to $T^*\mathbb{C}P^1$, s.t. $t \cdot \omega_{\mathbb{C}} = t\omega_{\mathbb{C}}$
- $\pi^{-1}(0) = \mathbb{C}P^1$ Lagrangian



The $\mathbb{R}-$picture

Conical symplectic resolution

A conical symplectic resolution (CSR) of **weight** $k \in \mathbb{N}$ is

- A projective $\mathbb{C}^*$-equivariant resolution,

$$\begin{array}{c} \mathbb{C}^* \overset{\varphi}{\curvearrowright} \mathfrak{M} \\ \pi \downarrow \\ \mathbb{C}^* \overset{\varphi}{\curvearrowright} \mathfrak{M}_0 \end{array}$$

- $\mathfrak{M}_0$ normal affine holo^c Poisson variety whose $\mathbb{C}^*$-action contracts to a single fixed point:

$$\forall x \in \mathfrak{M}_0, \quad \lim_{t \to 0} t \cdot x = x_0,$$

  Such actions we call **conical**.
- $(\mathfrak{M}, \omega_{\mathbb{C}})$ holo^c symplectic, $t \cdot \omega_{\mathbb{C}} = t^k \omega_{\mathbb{C}}$.

Examples of conical symplectic resolutions

- Resolutions of Du Val singularities
- Hilbert schemes of points on them
- Nakajima quiver varieties
- Springer resolutions, resolutions of Slodowy varieties
- Hypertoric varieties
- Slices in affine Grassmanians
- Higgs/Coulomb branches of moduli spaces
  (3d Gauge theories with $\mathcal{N} = 4$ supersymmetry)
- All examples are complete hyperkähler manifolds.

Real sympectic structure on CSRs

- Def: An exact real symplectic manifold $(M, \omega = d\theta)$ is a **Liouville manifold** when

$$(M \setminus K, \theta) \cong (\Sigma \times [1, +\infty), R\alpha)$$

  where α is a positive contact form on Σ.
- Any CSR $(\mathfrak{M}, \varphi)$ is canonically a Liouville manifold $(\mathfrak{M}, \omega_{J,K})$, where $\omega_{\mathbb{C}} = \omega_J + i\omega_K$ and $\omega_{J,K} =$ any linear combo of ω_J, ω_K.
- Hence, the **compact** $\mathcal{F}(\mathfrak{M})$ and the **wrapped** $\mathcal{W}(\mathfrak{M})$ Fukaya categories are well-defined.
- We are interested in **closed exact Lagrangian submanifolds** of $(\mathfrak{M}, \omega_{J,K})$ ($L \subset \mathfrak{M}$ exact means $\theta_{|L}$ is exact)

Exact Lagrangians in CSRs

- When CSR $\pi : \mathfrak{M} \to \mathfrak{M}_0$ is of weight 1, its **core** $\mathfrak{L} = \pi^{-1}(0)$ is a complex Lagrangian subvariety.
- Otherwise **not**, e.g. $\mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2)$
- $\mathfrak{L} = \cup_{\alpha \in A} \mathfrak{L}_\alpha$
- If $\mathfrak{L}_\alpha$ smooth, $\mathfrak{L}_\alpha$ is exact.
- All $\mathfrak{L}_\alpha$ are non-isotopic.

Theorem (Ž.'19)

Any weight-1 CSR $\mathfrak{M}$ has at least $N \geq 1$ smooth core components, hence non-isotopic exact Lagrangians. Here N is the number of different (commuting) conical weight-1 $\mathbb{C}^*$-actions on $\mathfrak{M}$.

- We call the these **minimal components** of the core.
- Example: Du Val resolutions of type A: $\widetilde{\mathbb{C}^2/\mathbb{Z}/n} \to \mathbb{C}^2/\mathbb{Z}/n$ The core is A_{n-1} tree of spheres and they are all minimal.

Floer theory of minimal components

- Fukaya category $\mathcal{F}(\mathfrak{M})$
- objects: closed exact Lagrangian submanifolds
- morphisms: $Mor(L_1, L_2) = CF^*(L_1, L_2)$
  cohomologically: $HF^*(L_1, L_2)$

Proposition

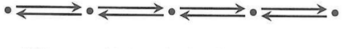
1. *Given a weight-1 CSR $\mathfrak{M}$, its minimal components are exact Lagrangians, hence $HF^*(\mathfrak{L}_{min}, \mathfrak{L}_{min}) \cong H^*(\mathfrak{L}_{min})$ for each minimal $\mathfrak{L}_{min}$.*
2. *For each pair $\mathfrak{L}^1_{min}, \mathfrak{L}^2_{min}$ of minimal components we have $HF^*(\mathfrak{L}^1_{min}, \mathfrak{L}^2_{min}) \cong H^*(\mathfrak{L}^1_{min} \cap \mathfrak{L}^2_{min})$.*
3. *Given a triple $\mathfrak{L}^1_{min}, \mathfrak{L}^2_{min}, \mathfrak{L}^3_{min}$ of minimal components, The Floer product*

$$HF^*(\mathfrak{L}^2_{min}, \mathfrak{L}^3_{min}) \otimes HF^*(\mathfrak{L}^1_{min}, \mathfrak{L}^2_{min}) \to HF^*(\mathfrak{L}^1_{min}, \mathfrak{L}^3_{min})$$

is isomorphic to the convolution product.

Representations of a double quiver

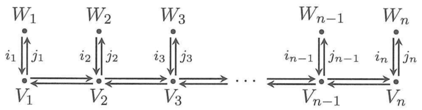
- Graph $Q = (I, E) \rightsquigarrow$ **double quiver** $Q^{\#} = (I, H := E \sqcup \bar{E})$



Double quiver of A_4

- The space of **Framed representations of double quiver**

  $M(Q, V, W) = \oplus_{h \in H}\mathrm{Hom}(V_{s(h)}, V_{t(h)}) \oplus_{i \in I}\mathrm{Hom}(V_i, W_i) \oplus_{i \in I}\mathrm{Hom}(W_i, V_i)$



- $GL(V) = \prod_{i \in I} GL(V_i) \curvearrowright M(Q, V, W)$ by conjugation.

Quiver varieties

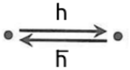
- $GL(V) = \prod_{i \in I} GL(V_i) \curvearrowright M(Q, V, W)$ by conjugation.
- Moment map $\mu : M(Q, V, W) \to \mathfrak{gl}(V)^*$
- **Nakajima quiver varieties**
  $\mathfrak{M}_\theta(Q, V, W) := \mu^{-1}(0)^{\theta-ss} / GL(V)$ smooth
  $\mathfrak{M}_0(Q, V, W) := \mu^{-1}(0) /\!\!/ GL(V)$ affine singular
- Depends only on dimensions $\mathbf{v} = \dim V, \mathbf{w} = \dim W$, so denote by $\mathfrak{M}_\theta(Q, \mathbf{v}, \mathbf{w}), \mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$
- There is a symplectic resolution
  $\pi : \mathfrak{M}_\theta(Q, \mathbf{v}, \mathbf{w}) \twoheadrightarrow \mathfrak{M}_1(Q, \mathbf{v}, \mathbf{w}) \subset \mathfrak{M}_0(Q, \mathbf{v}, \mathbf{w})$
- *N*akajima defines a conical weight-1 $\mathbb{C}^*$-action which makes it into a CSR.

Nakajima actions

- Recall the framed repn space of a double quiver $Q^{\#} = (I, H)$

$$M(Q, V, W) = \oplus_{h\in H}\mathrm{Hom}(V_{s(h)}, V_{t(h)})\oplus_{i\in I}\mathrm{Hom}(V_i, W_i)\oplus_{i\in I}\mathrm{Hom}(W_i, V_i)$$

- To construct a quiver variety, one has to pick a split $H = \Omega_0 \sqcup \overline{\Omega_0}$



- That makes $M(Q, V, W) = T^*R(\Omega_0, V, W)$, where $R(\Omega_0, V, W) = \oplus_{h\in\Omega_0}\mathrm{Hom}(V_{s(h)}, V_{t(h)}) \oplus \mathrm{Hom}(W_i, V_i)$
- Acting by $\mathbb{C}^*$ on fibres yields a weight-1 $\mathbb{C}^*$-action on $\mathfrak{M}_\theta(Q, \mathbf{v}, \mathbf{w}) \twoheadrightarrow \mathfrak{M}_1(Q, \mathbf{v}, \mathbf{w})$.
- We generalize this by using the other partitions $H = \Omega \sqcup \overline{\Omega}$, and get a family of actions which we call **Nakajima actions.**

Nakajima actions in type A

- By definition 2^{Q_1}, though not all are different.
- Use the description of coordinate ring $\mathbb{C}[\mathfrak{M}_0(\mathbf{v}, \mathbf{w})]$ by [Lusztig, Maffei]
- For $\mathbf{v} > 0$, get

$$N(\mathbf{w}) := \prod_{k=1}^{m-1} (s_{k+1} - s_k + 1),$$

where s_k are poisitons where $w_k \neq 0$.

- for general **dominant v**, get

$$N(\mathbf{v}, \mathbf{w}) := \prod_{i=1}^{k} N(\mathbf{w}^1) \cdots N(\mathbf{w}^k),$$

where $\mathbf{w} = \mathbf{w}^1 \sqcup \mathbf{w}^2 \cdots \sqcup \mathbf{w}^k$ is divided by the support of $\mathbf{v}$.

Nakajima actions in type A

- For arbitrary $\mathbf{v}$ use the LMN isomorphisms = Nakajima reflection functors,

$$\Phi_\sigma : \mathfrak{M}_\theta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{\sigma\cdot\theta}(\sigma *_{\mathbf{w}} \mathbf{v}, \mathbf{w})$$

  to pass from arbitrary $\mathbf{v}$ to a dominant vector $\mathbf{v}'$.
- By [Bezrukavnikov-Losev] Φ_σ intertwines Nakajima actions on both sides.

Theorem (Ž.'19)

Given a quiver variety $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$ of type A there is exactly $N(\mathbf{v}', \mathbf{w})$ different Nakajima actions, hence the same number of minimal components in its core $\mathfrak{L}_\theta(\mathbf{v}, \mathbf{w})$. Here $\mathbf{v}'$ is the associated dominant vector to $\mathbf{v}$.

- Dominant vector $\mathbf{v}'$, easily computable, hence $N(\mathbf{v}', \mathbf{w})$ as well.

Twisted full actions

- **Full quiver** weight-2 $\mathbb{C}^*$-action, acts on the *whole* $M(Q, V, W) = T^* R(Q_0, V, W)$
- $GL(\mathbf{w}) \curvearrowright \mathfrak{M}(Q, \mathbf{v}, \mathbf{w})$ symplectially by conjugations.
- **Twisted full actions** := 1-PS $\mathbb{C}^* \leq GL(\mathbf{w})$ combined with the full quiver action.
- Get a family of weight-2 actions, we count the even and conical ones.

Proposition (Ž.'20)

On a quiver variety $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$ of type A, Nakajima actions are exactly the square-roots of even and conical twisted full actions.

- Expect these to give all minimal components, i.e. $GL(\mathbf{w}) = Symp_{\mathbb{C}^*}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))^\circ$

Springer theory basics

- An important branch of GRT
- Classical results: Representations of Weyl groups [Springer, Kazhdan-Lusztig], representations of $U(\mathfrak{sl}_N)$ [Ginzburg].
- Central object: Springer resolution

$$\begin{array}{rl} T^*\mathcal{B} & \{(F,e) \mid F \in \mathcal{B},\ e \in \mathfrak{sl}_n,\ eF_i \subset F_{i-1}\} \\ \nu \downarrow & \quad \downarrow \\ \mathfrak{sl}_n \supset \mathcal{N} & \quad e \end{array}$$

- Generalized Springer resolution $T^*\mathcal{B}_p \xrightarrow{\nu_p} \overline{\mathcal{O}_{\tilde{p}^*}}$
- Generalized Springer fibre $\mathcal{B}_p^\lambda := \nu_p^{-1}(e_\lambda)$
- $Irr(\mathcal{B}_p^\lambda)$ parametrized by Standard Young tableaux $\mathbf{Std}_p^\lambda$
- (Non)smoothness and of components of $\mathcal{B}^\lambda$ is well-known [Pagnon-Ressayre, Barchini-Graham-Zierau, Fresse-Melnikov]
- **Not known:** (Non)smoothness of components of $\mathcal{B}_\mu^\lambda$

Slodowy varieties

- Given a nilpotent $e \in \mathfrak{sl}_n$ there is an $\mathfrak{sl}_2$-triple (e, f, h).
- Slodowy slice $S_e := e + ker(\mathrm{ad} f) \subset \mathfrak{sl}_n$
- Slodowy variety $\mathcal{S}_{e,p} := S_e \cap \overline{\mathcal{O}}_{p_+^*}$
- Restriction of Springer resolution yields a resolution $\widetilde{\mathcal{S}}_{e,p} := \nu_p^{-1}(\mathcal{S}_{e,p}) \to \mathcal{S}_{e,p}$.
- There is the **Kazhdan** $\mathbb{C}^*$**-action** $t \cdot x = t^2 \mathrm{Ad}(t^{-h})x$ on S_e, hence on $\mathcal{S}_{e,p}$ and $\widetilde{\mathcal{S}}_{e,p}$.
- It makes $\nu_p : \widetilde{\mathcal{S}}_{e,p} \to \mathcal{S}_{e,p}$ into a weight-2 CSR, whose core is $\mathcal{B}_p^\lambda$.
- Thus, its minimal components are *smooth components of* $\mathcal{B}_p^\lambda$.

Twisted Kazhdan actions

- $\nu_p : \widetilde{\mathcal{S}}_{e,p} \to \mathcal{S}_{e,p}$ is a weight-2 CSR with Kazhdan $\mathbb{C}^*$-action.
- $Z_e := C_{GL_n}(e, f, h)$ acts equivariantly on ν_p and symplectically on $\widetilde{\mathcal{S}}_{e,p}$.
- **Twisted Kazhdan actions** $:=$ 1-PS $\mathbb{C}^* \leq Z_e$ combined with the Kazhdan action
- Search the even and conical ones, as their square-roots are weight-1 conical.

Theorem (Ž.'20)

Given a nilpotent e, define $\mathbf{w}$ *by* $\lambda(e) = 1^{w_1} 2^{w_2} \dots n^{w_n}$. *Then*

- $Z_e \cong GL(\mathbf{w})$
- *There is exactly* $N(\mathbf{w})$ *different even and conical twisted Kazhdan actions on* S_e.
- *The same holds for* $\mathcal{S}_e = S_e \cap \mathcal{N}$ *(here* $p = (1, \dots, 1)$*).*
- *Thus, there is* $N(\mathbf{w})$ *minimal components in* $\mathcal{B}^\lambda$.

Towards the Maffei isomorphism

- For general p, some of these $N(\mathbf{w})$ actions on $\nu_p : \widetilde{\mathcal{S}}_{e,p} \to \mathcal{S}_{e,p}$ may overplap.
- Compare with quiver varieties by Maffei isomorphism:

$$\begin{array}{ccc} \mathfrak{M}(\mathbf{v},\mathbf{w}) & \xrightarrow{\widetilde{\varphi}} & \widetilde{\mathcal{S}}_{e,p} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \nu_p} \\ \mathfrak{M}^1(\mathbf{v},\mathbf{w}) & \xrightarrow{\varphi_1} & \mathcal{S}_{e,p} \end{array}$$

  where $\mathbf{w} - C\mathbf{v} = \mu = (p_1 - p_2, \ldots, p_n - p_{n+1})$.
- Expect (work in progress) φ and φ_1 to be equivariant with respect to $\mathbb{C}^* \times GL(\mathbf{w})$-action, where $GL(\mathbf{w}) \cong Z_e$ explicit.
- That would yield $N(\mathbf{v}', \mathbf{w})$ smooth components in $\mathcal{B}_p^\lambda$.

Further research - crystal operators

- There are certain **crystal operators** that interchange between irreducible components of diferent cores.
- For quiver varieties, founded by [Nakajima, Saito].
- Later, [Savage] translates via Maffei isomorphism to Springer fibres.
- Get maps

$$\widetilde{E_k} : Irr(\mathcal{B}_p^\lambda) \to Irr(\mathcal{B}_{p^-}^\lambda)$$

$$\widetilde{F_k} : Irr(\mathcal{B}_p^\lambda) \to Irr(\mathcal{B}_{p^+}^\lambda)$$

  where $p^{k,\pm} = (p_1, \ldots, p_{k-1}, p_k \pm 1, p_{k+1} \mp 1, p_{k+2}, \ldots, p_n)$.
- Using these maps and minimal components, one could generate many more smooth components of $\mathcal{B}_p^\lambda$ (work in progress).

The end

Thank you for listening.