Fourier optimization and the least quadratic non-residue

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CRG L-functions seminar
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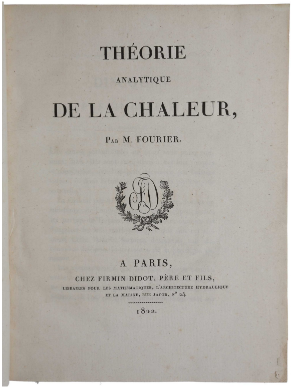
Fourier

Joseph Fourier
1768 - 1830

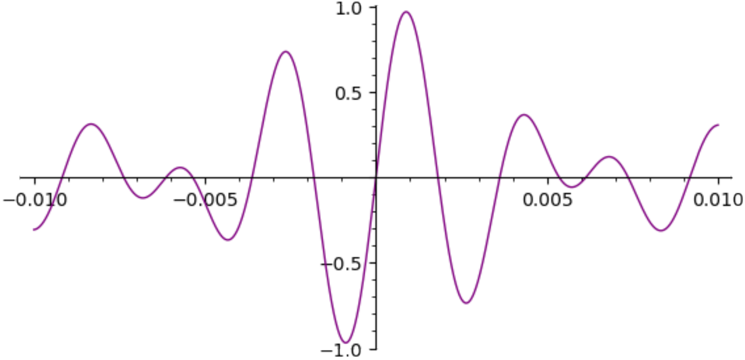


THÉORIE
ANALYTIQUE
DE LA CHALEUR,
PAR M. FOURIER.

A PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS,
LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE
ET LA MARINE, RUE JACOB, N° 24.
1822.

1822

Oscillations: "A Major" chord



$$f(x) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{2\pi i x k}, \qquad e^{2\pi i x k} = \cos(2\pi x k) + i \sin(2\pi x k).$$

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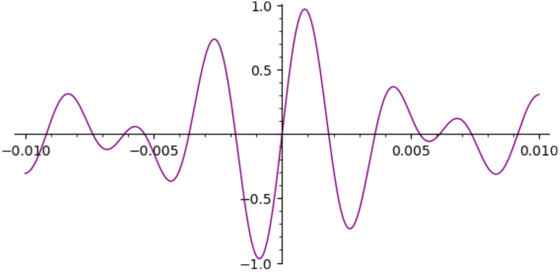
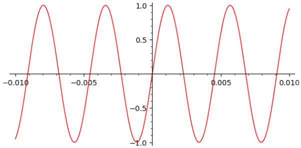
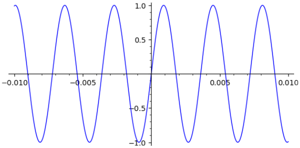
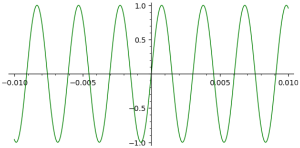




A: 440 Hz
$f(x) = \sin(440\pi x)$



C#: 550 Hz
$f(x) = \sin(550\pi x)$



E: 660 Hz
$f(x) = \sin(660\pi x)$

Fourier transform

- Let $f \in L^1(\mathbb{R})$. We define

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \cdot \xi} f(x) \, \mathrm{d}x \qquad (\xi \in \mathbb{R}).$$

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Fourier uncertainty: "the mass of a function and its Fourier transform cannot both be concentrated near the origin"

- Heisenberg: the mass of f and $\widehat{f}$ cannot be arbitrarily concentrated near the origin

$$||f||_2^2 \leq 4\pi |||x|f||_2 \cdot |||y|\widehat{f}||_2$$

Fourier extremal problems

- Impose conditions on f, $\widehat{f}$ and optimize some quantity

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$$\mathcal{A} := \inf \int_{\mathbb{R}} |f(x)| \, \mathrm{d}x,$$

subject to the conditions $f(0) \geq 1$, $f(x) \geq 0$, f is continuous, and $\operatorname{supp} \widehat{f} \subset [-1, 1]$

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- $\mathcal{A} = 1$, and

$$\left(\frac{\sin(\pi x)}{\pi x} \right)^2$$

is the only extremizer

Monotone one-delta problem

Joint with A. Chirre, D. K. Dimitrov, and M. Sousa

Problem

Find

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Theorem

- *There exists an even extremizer*
- $1.2750 < \mathcal{A}_1 < 1.2772$

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Conjecture

$$\mathcal{A}_1 = 1.277135042...$$

A Fourier extremal problem

Extremal Problem (EP)

Given $0 \leq A < \infty$, find

$$\mathcal{C}(A) := \sup_{0 \neq F \in \mathcal{A}} \frac{2\pi \left(\int_{-\infty}^{0} \hat{F}(t) e^{\pi t} dt - \int_0^\infty \hat{F}_-(t) e^{\pi t} dt - A \int_0^\infty \hat{F}_+(t) e^{\pi t} dt \right)}{||F||_1}$$

where the supremum is taken over the class of functions

$$\mathcal{A} = \{F : \mathbb{R} \to \mathbb{C}; F \in L^1(\mathbb{R}), \hat{F} \text{ is real-valued}\}.$$

Results concerning the EP

Theorem

Concerning the EP defined above:

1. *The supremum can be taken over $F \in \mathcal{A}$ with $\hat{F} \in C_c^\infty$*
2. *One has the endpoint values $\mathcal{C}(0) = 2$ and $\lim_{A\to\infty} \mathcal{C}(A) = 1$*
3. *The function $A \mapsto \mathcal{C}(A)$ is continuous and non-increasing in A.*

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- Hard to find exact value of $\mathcal{C}(A)$!
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Theorem

We have the following estimates:

1. $1.14599 < \mathcal{C}(1) < 1.14744$
2. $1.06079 < \mathcal{C}(3) < 1.06249$

Least Quadratic Nonresidue: Background I

Fix q prime. We consider the group of reduced residues $(\mathbb{Z}/q\mathbb{Z})^*$.

Definition

The number $a \in \mathbb{N}$ is called a *quadratic residue* if $\gcd(a, q) = 1$ and there exists $x \in N$ with $\gcd(x, q) = 1$ and $a \equiv x^2 \mod q$. It is called a *quadratic nonresidue* if $\gcd(a, q) = 1$ but no such x exists.

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Of course 1 is always a quadratic residue. The more interesting question is what is the first nonresidue:

Problem

Given q prime, if we define the *least quadratic nonresidue mod q* to be

$$n_q := \min\{a \in \{1, 2, ..., q-1\} \,|\, a \text{ is a quadratic nonresidue.}\}$$

how large can it be?

Background II

Recall that if q is a prime, we have the Legendre symbol defined on integers n

$$\chi(n) = \begin{cases} 1 \text{ if } n \text{ is a quadratic residue mod q} \\ -1 \text{ if } n \text{ is a quadratic nonresidue mod q} \\ 0 \text{ otherwise.} \end{cases}$$

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Moreover, the Legendre symbol has two important properties:

- It is totally multiplicative: for $m, n \in \mathbb{Z}$ we have $\chi(mn) = \chi(m)\chi(n)$
- It is q-periodic: for $n, k \in \mathbb{Z}$ we have $\chi(n + kq) = \chi(n)$

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i.e χ is a *Dirichlet character* modulo q.

Background III

With this function, we can make a first elementary observation:

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Proof.

Since $n_q \neq 1$ is a non-empty product of primes $p_1^{\alpha_1}...p_k^{\alpha_k}$.

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Proof.

Since $n_q \neq 1$ is a non-empty product of primes $p_1^{\alpha_1}...p_k^{\alpha_k}$. Since the Legendre symbol is totally multiplicative:

$$-1 = \chi(n_q) = \chi(p_1)^{\alpha_1}...\chi(p_k)^{\alpha_k}$$

So there must be an α_i odd, and p_i prime for which $\chi(p_i) = -1$. By minimality, it follows that $n_q = p_i$, i.e., n_q is prime. □

In fact, for any prime p there will be a q such that $n_q = p$. So n_q can be arbitrarily big, but is there a relation between the sizes of q and n_q?

History of the problem

The first serious study of this quantity is due to Vinogradov who in 1918 managed to prove

$$n_q = O(q^{\frac{1}{2\sqrt{e}}} \log^2 q).$$

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In 1952, Ankeny managed to prove that under the Generalized Riemann Hypothesis (GRH) we can get something better than what Vinogradov originally conjectured:

$$n_q = O(\log^2 q)$$

LQNR: The main result

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- Best (asymptotic) value: $C = 0.794$ (Lamzouri, Li, Soundararajan, 2016).

Theorem (Carneiro, Milinovich, QH., Ramos)

Assuming GRH,

$$n_q \leq \left(\frac{1}{\mathcal{C}(1)^2} + o(1) \right) \log^2 q$$

as $q \to \infty$.

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- Strategy: Guinand-Weil for the Dirichlet L-function

$$L(s, \chi) := \sum^{\infty} \frac{\chi(n)}{n^s}.$$

The explicit formula

Lemma (Guinand–Weil Explicit Formula)

Assume GRH. For “nice", even, real-valued h with $\widehat{h}(0) = 0$,

$$\sum_{\gamma_\chi} h(\gamma_\chi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \,\mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{2-\chi(-1)}{4} + i\frac{u}{2}\right) du$$
$$- \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \chi(n) \hat{h}\left(\frac{\log n}{2\pi}\right).$$

where the sum on the left-hand side runs over the ordinates of the non-trivial zeros of $L(s,\chi)$ and $\Lambda(n)$ is the von Mangoldt function defined to be $\log p$ if $n = p^k$, p a prime and $k \geq 1$, and zero otherwise.

Asymptotic Analysis I

Let's look at the sum over primes first!

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$$\frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)\chi(n)}{\sqrt{n}}\hat{h}\left(\frac{\log n}{2\pi}\right) = \frac{1}{\pi}\sum_{n=2}^{n_q-1} + \sum_{n=n_q}^{\infty}\frac{\Lambda(n)\chi(n)}{\sqrt{n}}\hat{h}\left(\frac{\log n}{2\pi}\right)$$

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and now notice $\chi(n)=1$ for all $n<n_q$, by the definition of the LQNR, so that the above is

$$\geq\frac{1}{\pi}\sum_{n=2}^{n_q-1}\frac{\Lambda(n)}{\sqrt{n}}\hat{h}\left(\frac{\log n}{2\pi}\right)-\frac{1}{\pi}\sum_{n=n_q}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left|\hat{h}\left(\frac{\log n}{2\pi}\right)\right|$$

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which by the Prime Number Theorem is equal to

$$=\sqrt{n_q}\int_{-\infty}^{0}\hat{F}(y)e^{\pi y}dy-\sqrt{n_q}\int_{0}^{\infty}|\hat{F}(y)|e^{\pi y}dy+O((\log\log q)^2)$$

Asymptotic Analysis II

Moreover, by Stirling's formula $\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|}\right)$ for $|s| > 1$, so

$$\left|\int_{-\infty}^{\infty} h(u)\, \mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{a_\chi}{2} + i\frac{u}{2}\right) du\right| \leq \int_{-\infty}^{\infty} |h(u)| \log(2+|u|) du = O(1)$$

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Finally, the sum over zeros is bounded by

$$\begin{aligned}\left|\sum_{\gamma_\chi} h(\gamma_\chi)\right| &\leq \sum_{\gamma_\chi} |F(\gamma_\chi)| \\ &= \int_{-\infty}^{\infty} |F(t)| dN(t,\chi) \\ &\leq \frac{\log q}{2\pi} ||F||_1 + O\left(\frac{\log q}{\log\log q}\right)\end{aligned}$$

again using Stieltjes integration, where $N(T,\chi)$ is the number of zeros $\beta + i\gamma$ of $L(s,\chi)$ with $0 < \beta < 1$ and $0 \leq \gamma \leq T$.

Conclusion: Obtaining the Extremal Problem

From our inequalities for the LHS and the RHS, we can say

$$\sqrt{n_q}\int_{-\infty}^{0}\hat{F}(y)e^{\pi y}dy-\sqrt{n_q}\int_{0}^{\infty}|\hat{F}(y)|e^{\pi y}dy \leq \frac{\log q}{2\pi}||F||_1+O\left(\frac{\log q}{\log\log q}\right)$$

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Taking the infimum over the class of admissible functions, we have

$$n_q\leq\left(\frac{1}{\mathcal{C}(1)^2}+o(1)\right)\log^2 q$$

History and Background I

For $q \in \mathbb{N}$ and $1 \leq a \leq q$ with $\gcd(a, q) = 1$, we consider the arithmetic progression

$$a, a+q, a+2q, ..., a+kq, ...$$

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A natural follow-up question is the following:

Problem

If q, a are as above, and we define the *least prime in the arithmetic progression* $\equiv a \bmod q$ to be

$$P(a,q) := \min\{a+kq \mid k \in \mathbb{Z}, a+kq \text{ prime}\}$$

how large can it be?

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- Explicit L: Pan, Chen, Jutila, Graham, Wang, and Heath-Brown...
- Xylouris: $L = 5$
- Conjecture: $P(a,q) \leq C_\varepsilon q^{1+\varepsilon}$.

Assuming GRH, in 1996 Bach and Sorensen showed that as $q \to \infty$

$$P(a,q) \leq (1 + o(1))(\phi(q)\log q)^2$$

LPAP: The main result

This was also refined by Lamzouri, Li, and Soudararajan, who proved that

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Theorem (Carneiro, Milinovich, QH., Ramos)

Conditionally on the GRH,

$$P(a,q) \leq \left(\frac{1}{\mathcal{C}(3)^2} + o(1)\right)(\phi(q)\log q)^2$$

as $q \to \infty$.

since $\frac{1}{\mathcal{C}(3)^2} < 0.8887$.

LPAP: Outline of the Strategy

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- Key ingredient 1: cancellation property of Dirichlet characters (orthogonality)

$$\frac{1}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi(a)}\chi(n) = \begin{cases} 1 \text{ if } n \equiv a \bmod q \\ 0 \text{ otherwise} \end{cases}$$

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- Key ingredient 2: Brun-Titschmarch inequality (primes in short intervals)

$$\#\{p \text{ prime} : x < p \leq x + y, p \equiv a \bmod q\} \leq \frac{2y}{\phi(q) \log(y/q)}$$

(this is where the parameter $A = 3$ comes from).

Estimating the Sharp Constant: Lower Bound I

Examples! The expression in the numerator of our functional of interest

$$\int_{-\infty}^{0} \hat{F}(t)e^{\pi t}dt - \int_{0}^{\infty} \hat{F}_{-}(t)e^{\pi t}dt - A\int_{0}^{\infty} \hat{F}_{+}(t)e^{\pi t}dt$$

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suggests us that it is beneficial to take functions that are concentrated on the left side of origin.

Inspired by this intuition, we consider linear combinations of functions of the form $|x|^k e^{\pi x}\mathbf{1}_{\mathbb{R}_-}$, $k \in \mathbb{N}$, and we optimize over translations and dilations of such combinations. For example, in the case $A = 1$, a good approximant is $\hat{F}(x) = g(\frac{x-0.47}{0.42})$ where

$$g(x) = e^x \left(0.0006x^7 + 0.0005x^5 + x^3 + 0.0405x\right)(\mathrm{sgn}(x) - 1)$$

Estimating the Sharp Constant: Lower Bound II

which looks like

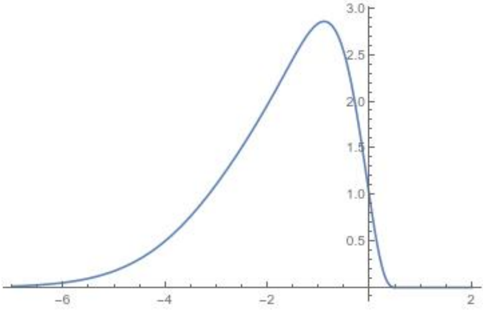

Figure: Plot of $f(x)$ for LQNR

and gives us

$$1.143 < \mathcal{C}(1)$$

Estimating the Sharp Constant: Lower Bound II

We do a similar process for $A = 3$, which gives us

$$g(x) = e^x \left(0.001x^7 - 0.00685x^5 + 1.x^3 - 0.0155x\right)(\text{sgn}(x) - 1.)$$

and we take $\hat{F}(x) = g\left(\frac{x-0.26}{0.32}\right)$ as our approximant

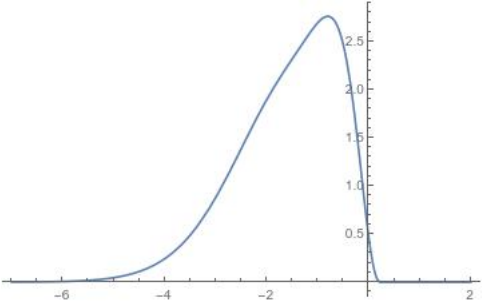

Figure: Plot of f(x) for LPAP

Estimating the Sharp Constant: Lower Bound III

This last function gives

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Estimating the Sharp Constant: Lower Bound III

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In the above, we were only looking for linear combinations of $|x|^k e^{\pi t} \mathbf{1}_{\mathbb{R}_-}$ when $k \leq 7$, but running this kind of procedure allowing x to be raised to even higher powers (k up to 23), we obtain the stated lower bounds:

1. $1.14599 < \mathcal{C}(1)$
2. $1.06079 < \mathcal{C}(3)$

Estimating the Sharp Constant: Upper Bound I

Recall that

$$\mathcal{C}(A) \triangleq \sup_{0 \neq F \in \mathcal{A}} \frac{2\pi \left(\int_{-\infty}^{0} \hat{F}(t) e^{\pi t} dt - \int_{0}^{\infty} \hat{F}_{-}(t) e^{\pi t} dt - A \int_{0}^{\infty} \hat{F}_{+}(t) e^{\pi t} dt \right)}{||F||_1}$$

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Fix $F \in \mathcal{A}$. For a function $\psi \in L^1(\mathbb{R}_+)$ with $-e^{\pi t} \leq \psi(t) \leq A e^{\pi t}$ when $t > 0$, we have by Fourier multiplication

$$\begin{aligned} &\int_{-\infty}^{0} \hat{F}(t) e^{\pi t} dt - \int_{0}^{\infty} \hat{F}_{-}(t) e^{\pi t} dt - A \int_{0}^{\infty} \hat{F}_{+}(t) e^{\pi t} dt \\ &\leq \int_{-\infty}^{\infty} \hat{F}(t) (\mathbf{1}_{\mathbb{R}_-}(t) e^{\pi t} - \mathbf{1}_{\mathbb{R}_+}(t) \psi(t)) dt \\ &= \int_{\mathbb{R}} F(x) (\widehat{\mathbf{1}_{\mathbb{R}_-} e^{\pi(\cdot)}} - \widehat{\mathbf{1}_{\mathbb{R}_+} \psi}) dx \\ &\leq ||F||_1 ||(\widehat{\mathbf{1}_{\mathbb{R}_-} e^{\pi(\cdot)}} - \widehat{\mathbf{1}_{\mathbb{R}_+} \psi})||_\infty \end{aligned}$$

Estimating the Sharp Constant: Upper Bound II

So one can find upper bounds looking for extremizers of the following dual problem:

Dual Extremal Problem (EP*)

Given $A \geq 0$, find

$$\mathcal{C}^*(A) := \inf_{\psi \in \mathcal{B}} 2\pi \, ||(\widehat{\mathbf{1}_{\mathbb{R}_-} e^{\pi(\cdot)}} - \widehat{\mathbf{1}_{\mathbb{R}_+} \psi})||_\infty$$

where $\mathcal{B} = \{\psi \in L^1(\mathbb{R}_+); -e^{\pi t} \leq \psi(t) \leq Ae^{\pi t} \text{ when } t > 0\}$

since, by the above, $\mathcal{C}(A) \leq \mathcal{C}^*(A)$.

Estimating the Sharp Constant: Upper Bound II

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For example, one can take the truncated function $\psi(t) = e^{\pi t}\mathbf{1}_{[0,T]}(t)$ ($A = 1$ for simplicity) which yields

$$||(\widehat{\mathbf{1}_{\mathbb{R}_-} e^{\pi(\cdot)}} - \widehat{\mathbf{1}_{\mathbb{R}_+} \psi})||_\infty = \sup_{x \in \mathbb{R}} \left| \frac{2 - e^{2\pi i x T} e^{\pi T}}{2\pi i x + \pi} \right|$$

Estimating the Sharp Constant: Upper Bound III

And we can calculate the sup on the RHS and optimize over T using techniques of standard calculus. Experimentally we have found it is advantageous to allow for changes of sign in ψ. Introducing a finite number of steps $0 = T_0 < T_1 < ... < T_N$, take

$$\psi(t) = \sum_{n=0}^{N-1} (-1)^n e^{\pi t} \mathbf{1}_{[T_n, T_{n+1}]}(t)$$

and optimize over the best choices of $T_0, ..., T_N$, whence we obtain the stated upper bounds for $A = 1, 3$:

1. $\mathcal{C}(1) < 1.14744$
2. $\mathcal{C}(3) < 1.06249$

Thank you!