Voronoi summation formula, their applications and other identities

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Definitions

Define $d(n)$ and $\lambda(n)$:

- $d(n) = \sum_{d|n} 1$
- $\lambda(n) = (-1)^{a_1+a_2..+a_k}$

where

$$n = p_1^{a_1} p_2^{a_2} ... p_k^{a_k}$$

Riemann Zeta function

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We have

$$\zeta(s) = \chi(s)\zeta(1-s)$$

where

$$\chi(s) = \pi^{s-1/2} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}.$$

Voronoi-type summation formulas

The Voronoi summation formula for $d(n)$

- The celebrated result of Voronoi associated with $d(n)$ is given by

$$\sum_{n\leq x}{}' d(n) = x(\log x + (2\gamma - 1)) + \frac{1}{4} + \sqrt{x}\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\left(-Y_1(4\pi\sqrt{nx}) - \frac{2}{\pi}K_1(4\pi\sqrt{nx})\right).$$

- Here, $Y_\nu(z)$ and $K_\nu(z)$ denote the Bessel and modified Bessel functions of the second kind of order ν respectively.

The Dirichlet Divisor problem

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Using Voronoi summation formula, Voronoi(1910) proved that

$$\Delta(x) = O(x^{1/3}\log x) \text{ as } x \to \infty.$$

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$$\sum_{\alpha \le n \le \beta}{}' d(n) f(n) = \int_{\alpha}^{\beta} (2\gamma + \log t) f(t) \, \mathrm{d}t + 2\pi \sum_{n=1}^{\infty} d(n) \int_{\alpha}^{\beta} f(t) \left(\frac{2}{\pi} K_0(4\pi\sqrt{nt}) - Y_0(4\pi\sqrt{nt}) \right) \mathrm{d}t.$$

where $f(t)$ is a function of bounded variation in (α, β) with $0 < \alpha < \beta$.

Let $\sigma_{-s}(n) = \sum_{d|n} d^{-s}$.
We have

$$\sum_{\alpha<n<\beta} \sigma_{-s}(n) f(n) = \int_{\alpha}^{\beta} (\zeta(1+s) + t^{-s}\zeta(1-s)) f(t) \mathrm{d}t$$

$$+ 2\pi \sum_{n=1}^{\infty} \sigma_{-s}(n) n^{s/2} \int_{1}^{\infty} t^{-s/2} f(t) \left\{ \left(\frac{2}{\pi} K_s(4\pi\sqrt{nt}) - Y_s(4\pi\sqrt{nt}) \right) \cos\left(\frac{\pi s}{2}\right) - J_s(4\pi\sqrt{nt}) \sin\left(\frac{\pi s}{2}\right) \right\} \mathrm{d}t.$$

For $\Re s > 1$,

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s}$$

Voronoi summation for $d^2(n)$

Theorem (C.-Dixit(2024))

Let $d(n)$ be the divisor function and $\phi(x)$ be a function satisfying some nice hypotheses, then assuming RH and the simplicity of the zeros of the zeta function, there exists a sequence of numbers $\{T_n\}_{n=1}^{\infty}$ with $T_n \to \infty$, such that

$$\sum_{n=1}^{\infty} d^2(n)\phi(n) = \frac{1}{\pi^8}\int_0^{\infty}(A_0 + A_1\log x + A_2\log^2 x + \pi^6\log^3 x)\phi(x)\mathrm{d}x$$
$$+ \lim_{T_n\to\infty}\sum_{|\gamma_m|\le T_n}\frac{\zeta^4\left(\frac{\rho_m}{2}\right)}{2\zeta'(\rho_m)}\int_0^{\infty}\phi(x)x^{\frac{\rho_m}{2}-1}\mathrm{d}x$$
$$+ \frac{\sqrt{2}}{\pi^2}\sum_{n=1}^{\infty} nc(n)\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\phi(z)}{z}\left(\frac{2}{\pi}K_0(4\sqrt{x}) - Y_0(4\sqrt{x})\right)$$
$$\left(\frac{2}{\pi}K_0(4\sqrt{y}) - Y_0(4\sqrt{y})\right)\cos\left(\frac{2\sqrt{nxy}}{z\pi}\right)\mathrm{d}z\mathrm{d}x\mathrm{d}y.$$

$c(n)$ is a multiplicative function defined as

$$c(p^k) = \binom{k+3}{6} - p\binom{k+1}{6}.$$

Previous Results:

Voronoi and later Cohen had found an interesting identity for $d(n)$

$$\sum_{n=1}^{\infty} d(n) \frac{x \log(x/n)}{x^2 - n^2} = \frac{\pi^2}{4} \log x - \frac{\pi^2}{2}\gamma + \frac{\log(4\pi^2 x)}{4x} + \sum_{n=1}^{\infty} d(n) K_0(4\pi\sqrt{nx}).$$

Cohen-Voronoi type Identity

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By replacing $x \to ix$ and $x \to -ix$ and adding and simplifying

$$\frac{x}{\pi} \sum_{n=1}^{\infty} \frac{d(n)}{x^2 + n^2} = -\frac{1}{2} \log x - \gamma - \frac{1}{4\pi x} + \sum_{n=1}^{\infty} d(n) \left(K_0(4\pi e^{i\pi/4}\sqrt{nx}) + K_0(4\pi e^{-i\pi/4}\sqrt{nx}) \right).$$

Cohen-type identity for $d^2(n)$

Theorem (C.-Dixit(2024))

$$\sum_{n=1}^{\infty} d^2(n)\frac{x\log^2(x/n)}{x^2-n^2}$$
$$= 8\sum_{n=1}^{\infty} c(n)\left(K_0(4\pi\sqrt{nx})K_1(4\pi\sqrt{nx}) - \sqrt{nx}\frac{K_0^2(4\pi\sqrt{nx})}{4\pi}\right)$$
$$+ 4R_1(x) + 4R_0(x) + \lim_{T_n\to\infty}\sum_{|\gamma_m|\leq T_n} R_{\rho_m}(x).$$

$$R_0(x) = C_2 \log^2 x + C_1 \log x + C_0.$$

$$R_1(x) = 4\pi^2 \frac{\log^2 x}{x}.$$

$$R_{\rho_m} = \frac{\zeta^4(\frac{\rho_m+1}{2})\Gamma^4(\frac{\rho_m+1}{2})}{\zeta'(\rho_m)\Gamma(\rho_m)}(4\pi^2 x)^{-\frac{1+\rho_m}{2}}.$$

Cohen for product of generalized divisor

$$\sum_{n=1}^{\infty} \sigma_a(n)\sigma_b(n) \frac{x\left(x^{-a} - n^{-a}\right)\left(x^{-b} - n^{-b}\right)}{x^2 - n^2}$$

$$= \pi^{a+b} 2^{2a+2b+2} \sin(\pi a/2) \sin(\pi b/2) \sum_{n=1}^{\infty} c(n)(16\pi^2 nx)^{(-a-b)/2}$$

$$\Bigg(K_{a-1}\left(4\pi\sqrt{nx}\right) K_b\left(4\pi\sqrt{nx}\right) + K_{b-1}\left(4\pi\sqrt{nx}\right) K_a\left(4\pi\sqrt{nx}\right)$$

$$+ \frac{(a+b-1)\sqrt{nx}}{4\pi} K_b\left(4\pi\sqrt{nx}\right) K_a\left(4\pi\sqrt{nx}\right)\Bigg) + R(x).$$

Voronoi summation for Liouville Lambda function

Theorem (C.-Dixit(2024))

Let $\lambda(n)$ be the Liouville Lambda function and $\phi(x)$ be a function satisfying certain nice hypothesis, then there exists a sequence of numbers $\{T_n\}_{n=0}^{\infty}$ with $T_n \to \infty$, such that

$$\sum_{n=1}^{\infty} \lambda(n)\phi(n) = \frac{1}{2\zeta(\frac{1}{2})} \int_0^{\infty} \frac{\phi(x)}{\sqrt{x}} \mathrm{d}x + \lim_{T_n \to \infty} \sum_{|\gamma_m| < T_n} \frac{\zeta(2\rho_m)}{\zeta'(\rho_m)} \int_0^{\infty} \phi(x) x^{\rho_m - 1} \mathrm{d}x + \frac{1}{\sqrt{2\pi}} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{\mu(q)}{\sqrt{q}} \int_0^{\infty} \frac{\phi(x)}{\sqrt{x}} \sin\left(\frac{p^2 q x}{2\pi} + \frac{\pi}{4}\right) \mathrm{d}x,$$

where $\mu(n)$ is the Mobius function and $c(n) = p\mu(q)$ for $n = p^2 q$ and q is square-free.

Cohen-type identity for Liouville Lambda

Theorem (C.-Dixit(2024))

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\lambda(n)\frac{n}{x^2+n^2} = \frac{x^{-1/2}}{2\pi\sqrt{2}}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu(q)}{\sqrt{q}}e^{\frac{-\pi x q p^2}{2}} + \frac{\zeta(0)}{2\zeta(1/2)\Gamma(1/2)}(2\pi x)^{-1/2} + \lim_{T_n\to\infty}\sum_{|\gamma_m|\leq T_n}\frac{\zeta(2\rho_m-1)\Gamma(2\rho_m-1)}{\zeta'(\rho_m)\Gamma(\rho_m)}(2\pi x)^{-\rho_m}.$$

Original Ramanujan-Guinand Identity

For $\alpha\beta = \pi^2$, we have

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-z}n^{z/2}K_{z/2}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-z}n^{z/2}K_{z/2}(2n\beta)$$
$$= \frac{1}{4}\Gamma\left(\frac{z}{2}\right)\zeta(z)(\beta^{\frac{1-z}{2}} - \alpha^{\frac{1-z}{2}}) + \frac{1}{4}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)(\beta^{\frac{1+z}{2}} - \alpha^{\frac{1+z}{2}}).$$

For $\alpha\beta = 1$

$$\frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}\lambda(n)n^2\alpha^2K_{1/2}(\pi n^2\alpha^2)$$
$$= \frac{\sqrt{2}}{16}\sum_{n=1}^{\infty}c(n)n^2\beta^3K_{1/2}\left(\frac{n^2\pi\beta^2}{8}\right)\left(K_{1/4}\left(\frac{n^2\pi\beta^2}{8}\right)+K_{3/4}\left(\frac{n^2\pi\beta^2}{8}\right)\right)$$
$$+\frac{\Gamma(1/2)}{2\zeta(1/2)\Gamma(1/4)}(\sqrt{\pi\alpha})^{-1/2}+\lim_{T_m\to\infty}\sum_{|\gamma_m|\leq T_n}\frac{\zeta(2\rho_m)\Gamma(\rho_m)}{\zeta'(\rho_m)\Gamma(\rho_m/2)}(\sqrt{\pi\alpha})^{-\rho_m}.$$

Potential Questions

- How can Voronoi summation formula for $d^2(n)$ be used for reducing the error term for $\sum_{n\leq x} d^2(n)$?
- How can we use Voronoi summation formula for $\lambda(n)$ to show infinitely many sign changes in $\sum_{n\leq x} \lambda(n)$?
- How can one use Cohen type formulas to derive the Voronoi summation formulas?

Thank You