Explicit bounds for ζ and a new zero-free region

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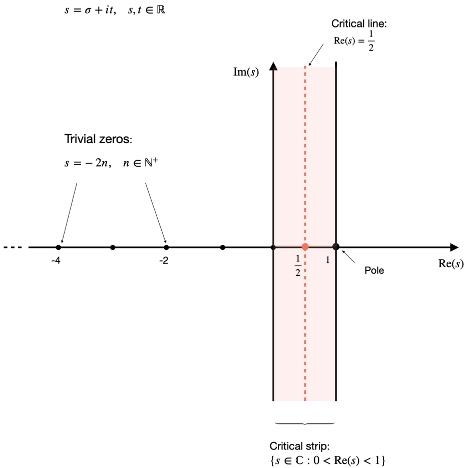
The Riemann zeta function

The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

for $\mathrm{Re}(s) > 1$, and its analytic continuation elsewhere.

Zeros of the Riemann zeta function



Zeros of ζ

Riemann Hypothesis

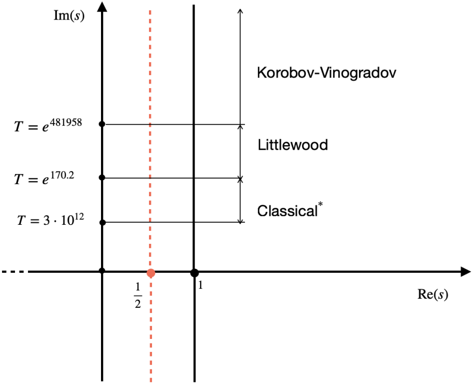
All the non-trivial zeros of ζ have real part equal to $\frac{1}{2}$.

- Partial verification of RH up to $T = 3 \cdot 10^{12}$ (Platt-Trudgian, 2021)

Main approaches to the problem

- Detecting zero-free regions for ζ
- Zero-density estimates

Zero-free regions for ζ



Zero-free regions for ζ

- Classical zero-free region (Mossinghoff-Trudgian-Yang, 2022)

$$\sigma \geq 1 - \frac{1}{5.558691 \log T}, \qquad T \geq 3$$

- Littlewood zero-free region (Yang, 2023)

$$\sigma \geq 1 - \frac{\log \log T}{21.233 \log T}.$$

- Korobov-Vinogradov zero-free region (B.)

$$\sigma \geq 1 - \frac{1}{53.989 \log^{2/3} T (\log \log T)^{1/3}}$$

Detecting Korobov-Vinogradov zero-free region

Main tool:

Sharp upper bound for $|\zeta(\sigma + it)|$ when σ sufficiently close to 1

- The proof follows (Ford, 2002)

Upper bounds on $\zeta(s)$

Sharpest upper bound when σ is close to 1 (Vinogradov, 1958)

$$|\zeta(\sigma + it)| \ll |t|^{B(1-\sigma)^{3/2}+\epsilon}$$

Explicit version (Richert, 1967)

$$|\zeta(\sigma + it)| \leq A|t|^{B(1-\sigma)^{3/2}} \log^{2/3}|t| \qquad |t| \geq 3,\ \frac{1}{2} \leq \sigma \leq 1$$

- Kulas, Cheng (1999)
- $A = 76.2,\ B = 4.45$ Ford (2002)
- $A = 70.6995,\ B = 4.43795$ B.

Statement of the new result

Theorem (B.)

The following estimate holds for every $|t| \geq 3$ and $\frac{1}{2} \leq \sigma \leq 1$:

$$|\zeta(\sigma + it)| \leq A|t|^{B(1-\sigma)^{3/2}} \log^{2/3} |t|,$$

with $A = 70.6995$ and $B = 4.43795$.

Idea of the proof

The proof follows (Ford, 2002)

1. Find upper bounds for the Vinogradov Integral
2. Estimate exponential sums
3. Bound $|\zeta(\sigma + it)|$

First step: the Vinogradov integral

The Vinogradov integral is defined as

$$J_{s,k}(P) = \int_{[0,1]^k} \left| \sum_{1 \le x \le P} e\left(\alpha_1 x + \cdots + \alpha_k x^k\right) \right|^{2s} d\boldsymbol{\alpha}$$

where $s, k \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ and $e(z) = e^{2\pi i z}$.

Equivalently, it is the number of solutions of the simultaneous equations

$$\sum_{i=1}^{s} \left(x_i^j - y_i^j\right) = 0 \quad (1 \le j \le k); \quad 1 \le x_i, y_i \le P.$$

Bounds for the Vinogradov integral

Non-explicit bounds-Conjecture

$$J_{s,k}(P) \ll_{s,k,\epsilon} \max\{P^{s+\epsilon}, P^{2s-\frac{1}{2}k(k+1)+\epsilon}\}, \qquad s,k \in \mathbb{N},\ \epsilon > 0$$

- $k = 3$ (Wooley, 2016), $k > 3$ (Bourgain, Demeter, Guth, 2016)

Explicit bounds

$$J_{s,k}(P) \leqslant D(s,k)P^{2s-k(k+1)/2+\eta(s,k)}, \qquad \eta(s,k) \geqslant 0$$

- $\eta(k,s) \approx \frac{3}{8}k^2 e^{1/2-2s/k^2}$ and $D(s,k) = O(k^{k^3})$ Ford (2002)
  - Improved for $k \geq 16000$ by Preobrazhenskii (2011)
- $\eta(k,s) \approx \frac{8}{25}k^2 e^{0.6494-2s/k^2}$ and $D(s,k) = O(k^{k^3})$ B.

Bounds for the Vinogradov integral

A further bound for $J_{s,k}(P)$

If $k \geq 129$, there is an integer $s \leq \rho k^2$ such that for $P \geq 1$,

$$J_{s,k}(P) \leq k^{\theta k^3} P^{2s - \frac{1}{2}k(k+1) + 0.001k^2},$$

where ρ, θ vary for different ranges of k.

- ρ increases in k
- θ decreases in k

Sketch of the proof

$129 \leq k \leq 499$	$500 \leq k < 90000$	$k \geq 90000$
- Computation - Improved Ford's argument	- Tyrina's method - Improved Ford's argument	- Preobrazhenskii's argument

The new tool is Tyrina's method

	Ford's recursive argument	B.'s recursive argument
Starting point	$\Delta_1 \leq \frac{1}{2}k^2(1 - \frac{1}{k})$	$\Delta_{n_0} \leq 0.4k^2, \quad n_0 = \lceil 0.1247k \rceil$

Incomplete systems

Given

$$\mathcal{C}(P,R) = \left\{ n \le P \mid \text{prime factors in } (\sqrt{R}, R] \right\},$$

we define

$$J_{s,k,h}(\mathcal{C}(P,R)) = \int_{[0,1]^t} |f(\boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha}$$

where

$$f(\boldsymbol{\alpha}) = f(\boldsymbol{\alpha}; P, R) = \sum_{x \in \mathcal{C}(P,R)} e\left(\alpha_h x^h + \cdots + \alpha_k x^k\right), \quad \boldsymbol{\alpha} = (\alpha_h, \cdots, \alpha_k).$$

Equivalently, they are the number of solutions of the simultaneous equations

$$\sum_{i=1}^{s} \left(x_i^j - y_i^j\right) = 0 \quad (h \le j \le k); \quad x_i, y_i \in \mathcal{C}(P,R).$$

Second step: exponential sum estimate

We define

$$S(N,t) := \max_{0<u\leq 1} \max_{N<R\leq 2N} \left| \sum_{N\leq n\leq R} \frac{1}{(n+u)^{it}} \right|$$

Estimate for $S(N,t)$

Suppose $N \geq 2$ is a positive integer, $N \leq t$ and set $\lambda = \frac{\log t}{\log N}$. Then

$$S(N,t) \leq 8.8 N^{1-1/\left(132.95\lambda^2\right)}.$$

Sketch of the proof

$1 \leq \lambda \leq 84$	$84 \leq \lambda \leq 220$	$\lambda \geq 220$
Ford's computation	Vinogradov's integral and incomplete systems	
	Computational	Optimise Ford's argument

- Critical point: $\lambda = 84$
- Choosing a splitting point for the last two intervals strictly greater than 220 would have negligible influence

Third step: Bound $|\zeta(\sigma + it)|$

$\frac{15}{16} \le \sigma \le 1,\ t \ge 10^{108}$	$\frac{15}{16} \le \sigma \le 1,\ 3 \le t \le 10^{108}$	$\frac{1}{2} \le \sigma \le \frac{15}{16},\ t \ge 3$
Apply lemma below with $C = 8.8, D = 132.95$	Ford's argument	

Lemma

Suppose $S(N,t) \le CN^{1-1/(D\lambda^2)}$, *where* $\lambda = \frac{\log t}{\log N}$ *and* $1 \le N \le t$. *Then, denoting* $B = \frac{2}{9}\sqrt{3D}$, *for* $\frac{15}{16} \le \sigma \le 1$, $t \ge 10^{100}$ *and* $0 < u \le 1$, *we have*

$$|\zeta(s)| \le \left(\frac{C + 1 + 10^{-80}}{\log^{2/3} t} + 1.569\, CD^{1/3} \right) t^{B(1-\sigma)^{3/2}} \log^{2/3} t.$$

Some consequences

- Improved Korobov-Vinogradov zero-free region
- Improved asymptotic zero-free region

$$\sigma \geq 1 - \frac{1}{48.0718(\log |t|)^{2/3}(\log \log |t|)^{1/3}}$$

- Improved estimate for the error term in the prime number theorem

$$\pi(x) - \mathrm{li}(x) \ll x \exp\left\{-d(\log x)^{3/5}(\log \log x)^{-1/5}\right\}, \quad d = 0.2125$$

Possible improvements

Considering the new exponential sum

$$\tilde{S}(N,t) := \max_{0<u\leq 1} \max_{N<R\leq mN} \left| \sum_{N\leq n\leq R} \frac{1}{(n+u)^{it}} \right|, \qquad 1 < m \leq 2,$$

would possibly improve A.

- The obstacle is $\lambda \leq 84$

For more details:

- arXiv:2306.10680

Thank you for your attention!