Branes, Quivers and BPS algebras

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Second PIMS Summer School on Algebraic Geometry in High-Energy Physics, August 23-27, 2021

2.16. Recapitulation

- We have argued that the derived category of coherent sheaves is a good model of branes and their bound states (see also beautiful lectures of Tudor).
- Morphisms in the brane category are in correspondence with massless string modes and can be encoded in a quiver diagram.

- The A_∞ structure capturing string interactions gives rise to the potential $W = \mathrm{Tr}\, B_1[B_2, B_3] + J B_3 I$
- The quiver diagram with potential in turn encodes a supersymmetric quantum mechanics describing the low-energy dynamics of the system of branes $A \to D0$.
- We are now going to look at the space of supersymmetric vacua of such a quiver quantum mechanics.

3. Supersymmetric vacua

3.1. Moduli of vacua

- Forgetting the potential, the Ω-backgroud, and the gauge group, the moduli space of vacua of our quantum mechanics would be computed in terms of de Rham cohomology of M. [Witten 1982]
- If we turn on the gauge group, the moduli space of vacua should be in correspondence with the de Rham cohomology of the quotient $M/GL(n)$ supplemented by the stability condition that requires (at least in our situation) the whole space $\mathbb{C}^n$ associated to the circular node to be generated by an action of B_i on I's, i.e.

$$\mathbb{C}^n = \sum_j \mathbb{C}[B_1, B_2, B_3] I_j$$

- For the purpose of our discussion, we label by $\mathcal{M}(n)$ the stable locus of M with a given choice of the framing and with the circular node of rank n. We then write $M(n) = \mathcal{M}(n)/GL(n)$.

3.2. Deformations of the differential

- If the potential W is non-trivial, the differential receives a correction proportional to $dW\wedge$.
- The main problem is non-compactness of $M(n)$. Luckily, we can introduce a deformation of the theory associated to flavor symmetries $U(n)$ of the system (Ω-background) that localizes the theory to fixed points of this symmetry.
- Physically, this can be done by introducing a vector multiplet associated to such a symmetry and turn on a non-zero vacuum expectation value for its scalars.
- The differential gets modified by $\sum_i \mu_i \iota_{X_i}$. See e.g. [Ohta-Sasai 2014].
- The resulting cohomology theory is known as de Rham model of equivariant critical cohomology. See e.g. the appendix of [MR-Soibelman-Yang-Zhao 1982].

3.3. Example of equivariant cohomology

- Just to gain some expecience, let me analyze a simple example of the equivariant cohomology
- We are going to compute the equivariant cohomology of $\mathbb{C}$ with the $U(1)$ action given by $e^{i\epsilon}z$ with $(z, \bar{z}) \in \mathbb{C}$ the complex coordinates.
- The differential is thus of the form

$$Q = dz\partial + d\bar{z}\bar{\partial} + \epsilon\iota_{z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}}$$

- Multiplication by dz and $d\bar{z}$ increases the degree of a form by one. $\iota_{z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}}$ decreases it by one. If we assign ϵ degree two, the differential Q is of degree one.
- When acting on a general form, the differential Q does not square to zero, e.g.

$$Q^2 z = Qdz = \epsilon z$$

  but restricting to $U(1)$ invariant forms, Q is nilpotent and its cohomology makes sense.

- At degree zero, we have

$$Qf(|z|^2) = (\bar{z}dz + zd\bar{z})\frac{\partial f(|z|^2)}{\partial |z|^2}$$

requiring f to be constant leading to one-dimensional cohomology.

- A general form at degree one is of the form

$$f(|z|^2)zd\bar{z} + g(|z|^2)\bar{z}dz$$

The kernel condition requires vanishing of

$$\left(\frac{\partial f(|z|^2)}{\partial |z|^2}|z|^2 - \frac{\partial g(|z|^2)}{\partial |z|^2}|z|^2 + f(|z|^2) - g(|z|^2)\right) dzd\bar{z}$$
$$+\epsilon \left(f(|z|^2)|z|^2 - g(|z|^2)|z|^2\right)$$

that implies $f(|z|^2) = g(|z|^2)$ but all such elements can be generated by the action of Q on degree-zero terms.

3.4. Borel localization theorem

- We could proceed with higher degrees and identify

$$H^*_{U(1)}(\mathbb{C}) = \mathbb{C}[\epsilon]$$

- Note that the cohomology has a single factor of $\mathbb{C}[\epsilon]$ and $\mathbb{C}$ has a single fixed point. This is not a coincidence!
- Acording to the Borel localization theorem, if X is a manifold with a $U(1)^m$ action and a finite set of fixed points $p_i \in F$, the embedding $\iota : F \hookrightarrow X$ induces an isomorphism

$$H^*_{U(1)^m}(X) \to H^*_{U(1)^m}(F) = \bigoplus_{i \in F} \mathbb{C}[\epsilon_1, \ldots, \epsilon_m] p_i$$

- Turning on the potential W, the Borel localization theorem holds as well, but we need to restrict to fixed points lying in the critical locus of the potential.
- The push-forward map ι_* then gives a fixed-point basis $\iota_* p_i$ of $H^*_{U(1)^m}(X)$ and we just need to find the fixed-point set.

3.6. D2-brane and $1d$ partitions

- Let us identify the fixed-point set for the D2-brane moduli. [Galakhov-Li-Yamazaki (2021), MR-Soibelman-Yang-Zhao (in progress)]
- Starting with the D2-brane superpotential

$$W = \mathrm{Tr}\,[B_1[B_2, B_3] + I(J_2 B_1 - J_1 B_2)]$$

we have the following equations of motion

$$\begin{aligned} [B_1, B_3] = IJ_1, \qquad [B_2, B_3] = IJ_2, \qquad [B_1, B_2] = 0 \\ B_1 I = 0, \qquad B_2 I = 0, \qquad J_2 B_1 - J_1 B_2 = 0 \end{aligned}$$

- It is straightforward to show that these conditions together with the stability condition require $J_1 = J_2 = 0$. This also implies that B_i mutually commute.
- We can then set $B_1 = B_2 = 0$ since

$$B_1 \mathbb{C}^n = B_1 \mathbb{C}[B_1, B_2, B_3] I = \mathbb{C}[B_1, B_2, B_3] B_1 I = 0$$

- We have thus identified the critical locus of W with a pair (B_3, I) subject to the stability condition

$$\mathbb{C}^n = \mathbb{C}[B_1]I$$

modulo gauge transformation

$$g : (B_1, I) \to (gB_1g^{-1}, gI)$$

- To gain some experience with finding fixed points, let us start with the analysis for $n = 1$. The value of I is non-vanishing due to stability. It can be thus set to 1 by the gauge transformation. The fixed-point condition then requires

$$e^{i\epsilon_1}B_1 = gB_1g^{-1} = B_1$$

leading to $B_1 = 0$. The only fixed point can be thus identified with the gauge orbit of $(B_1, I) = (0, 1)$

- Moving to $n = 2$, I being non-zero due to the stability condition and the gauge transformation allows us to fix

$$I = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- The residual gauge transformation allows to fix

$$B_1 = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}$$

- Let us now impose the fixed point condition

$$e^{i\epsilon_1} B_1 = e^{\epsilon_1} \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} = g B_1 g^{-1}$$

for g that now allows only rescaling of β. This leads to $\alpha = \gamma = 0$. Since $\beta \neq 0$ due to the stability, we can fix

$$(B_1, I) = \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

- The condition of (B_1, I) being at a fixed point requires an existence of g such that

$$e^{i\epsilon_1} B_1 = g B_1 g^{-1}$$

- Let us choose a basis for $\mathbb{C}^n$ that diagonalizes g. If a is a basis vectors with eigenvalue $e^{i(n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3)}$, we have

$$g B_1 a = g B_1 g^{-1} g a = e^{i((n_1+1)\epsilon_1+n_2\epsilon_2+n_3\epsilon_3)} B_1 a$$

  and $B_1 a$ is another basis vector with eigenvalue $e^{i((n_1+1)\epsilon_1+n_2\epsilon_2+n_3\epsilon_3)}$.

- Since I does not transfer under the $U(1)^3$ action, we get

$$I = gI$$

  and I is itself one of the eigenvectors.

- This produces a basis of $\mathbb{C}^n$ given by eigenvectors $B_1^n I$. In this basis, B_1 is obviously a nilponent matrix.

- For example for $n = 4$:

$$(B_1, I) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

- It will be convenient to draw the decomposition of the vector space as an equivariant complex

$$\mathbb{C}_0 \xrightarrow{B_1} \mathbb{C}_{\epsilon_1} \xrightarrow{B_1} \mathbb{C}_{2\epsilon_1} \xrightarrow{B_1} \mathbb{C}_{3\epsilon_1}$$

- Finally, let us visualize the weight decomposition of $\mathbb{C}^n$ as a row of n boxes. For example, in our case of $n = 4$,

3.7. D4-brane and $2d$ partitions

- We can proceed in a very same way in the case of the D4-brane framing. See e.g. lecture notes [Nakajima (1996)] for a $\mathbb{C}^2$ perspective or [MR-Soibelman-Yang-Zhao (2019)] for a $\mathbb{C}^3$ perspective.
- The system of equations following from the variation of the potential is now

$$\begin{aligned}[B_1, B_2] &= IJ \\ [B_1, B_3] &= [B_2, B_3] = 0 \\ JB_3 &= B_3 I = 0\end{aligned}$$

- From stability condition, we can see that $B_3 = 0$ reducing the system to the famous ADHM moduli.
- One can also show that the equations together with the stability condition require $J = 0$ and we are left with the system (B_1, B_2, I) satisfying the stability condition, B_1, B_2 mutually commuting and modulo

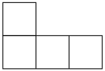
$$B_i \to g B_i g^{-1}, \qquad I \to gI$$

- (B_1, B_2, I) being a fixed point requires an existence of g such that

$$
\begin{aligned}
e^{i\epsilon_1} B_1 &= g B_1 g^{-1} \\
e^{i\epsilon_2} B_2 &= g B_2 g^{-1} \\
gI &= I
\end{aligned}
$$

- Let us pick a basis of $\mathbb{C}^n$ that diagonalizes g. If a is an eigenvecotor of g with eigenvalue $e^{i(n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3)}$, then $B_1 a$ is an eigenvector with eigenvalue $e^{i((n_1+1)\epsilon_1+n_2\epsilon_2+n_3\epsilon_3)}$ and $B_2 a$ is an eigenvector with eigenvalue $e^{i(n_1\epsilon_1+(n_2+1)\epsilon_2+n_3\epsilon_3)}$.
- Furthermore, since the whole $\mathbb{C}^n$ can be generated by an action of B_1, B_2 on I and these two mutually commute, we can see that the space $\mathbb{C}^n$ decomposes according to the $U(1)^3$ weights into subspaces specified by the Young diagram.

- For example



would be associated to the decomposition

$$
\begin{array}{ccccccc}
0 & \xrightarrow{B_1} & 0 & \xrightarrow{B_1} & 0 & \xrightarrow{B_1} & 0 \\
{\scriptstyle B_2}\uparrow & & {\scriptstyle B_2}\uparrow & & {\scriptstyle B_2}\uparrow & & {\scriptstyle B_2}\uparrow \\
\mathbb{C}_{\epsilon_2} & \xrightarrow{B_1} & 0 & \xrightarrow{B_1} & 0 & \xrightarrow{B_1} & 0 \\
{\scriptstyle B_2}\uparrow & & {\scriptstyle B_2}\uparrow & & {\scriptstyle B_2}\uparrow & & {\scriptstyle B_2}\uparrow \\
\mathbb{C}_0 & \xrightarrow{B_1} & \mathbb{C}_{\epsilon_1} & \xrightarrow{B_1} & \mathbb{C}_{2\epsilon_1} & \xrightarrow{B_1} & 0
\end{array}
$$

- It is easy to check that this corresponds to the gauge orbit of

$$
(B_1, B_2, I) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)
$$

3.8. D6-brane and $3d$ partitions

- In the case of D6-brane framing, we do not have any arrows going to the framing vertex and B_i's mutually commute.
- Decomposition of the vector space $\mathbb{C}^n$ into the eigenspace of g leads to the identification of fixed points with 3d partitions.
- For example, the 3d particion depicted in

corresponds to the gauge orbit of (B_1, B_2, B_3, I) equal to

$$\left(\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right)$$

3.9. The correspondence

- A crucial role in the construction is played by a correspondence $M(n+1,n)$ between $M(m+1)$ and $M(n)$, i.e. a closed subset $M(n+1,n)$ in $M(n)\times M(n+1)$.
- A point in $\mathcal{M}(n+1)\times\mathcal{M}(n)$ given by

$$\left(\left(B_1^{(1)},B_2^{(1)},B_3^{(1)},I^{(1)},J^{(1)}\right),\left(B_1^{(2)},B_2^{(2)},B_3^{(2)},I^{(2)},J^{(2)}\right)\right)$$

is in $\mathcal{M}(n+1,n)$ if there exists $\xi:\mathbb{C}^{n+1}\to\mathbb{C}^n$ satisfying

$$\xi B_i^{(1)}=B_i^{(2)}\xi,\qquad \xi I^{(1)}=I^{(2)},\qquad J_i^{(1)}=J_i^{(2)}\xi$$

[Nakajima (1994), Kontsevich-Soibelman (2010)]

- The stability implies that ξ is a surjective map and $S=\mathrm{Ker}\,\xi$ is a one-dimensional subspace of Ker $J^{(1)}$ that is invariant under the action of $B_i^{(1)}$.

- We can thus identify $\mathcal{M}(n+1, n)$ with an element of $\mathcal{M}(n+1)$ together with a choice of a $B_i^{(1)}$ invariant one-dimensional subspace $S \subset \text{Ker } J^{(1)}$.
- Using this description, we can quotient $\mathcal{M}(n+1, n)$ by the obvious action of $GL(n+1)$ and write

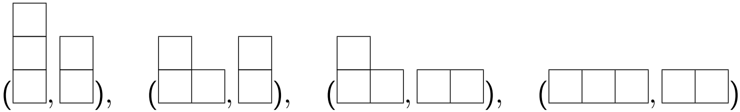
$$
\begin{array}{ccc}
 & M(n+1, n) & \\
 {}^{p}\swarrow & & \searrow^{q} \\
M(n+1) & & M(n)
\end{array}
$$

  where p is the obvious map forgetting the information about the subspace S and q is a quotient of $M(n+1)$ by S.
- Note also that $S = \text{Ker } \xi$ gives rise to a line bundle L on the correspondence called the tautological line bundle.

3.9. Fixed points of $M(n+1, n)$

- As we have seen above, fixed points of $M(n+1)$ are in correspondence with partitions of various dimensions containing $n+1$ boxes.
- In order to specify a point on $M(n+1, n)$, we need to further identify a subspace of $\mathbb{C}^{n+1}$ that is fixed under the action of $B_i^{(1)}$ and lies in the kernel of $J^{(1)}$.
- Since $J^{(1)} = 0$ in all three of our moduli spaces, we only require the subspace to be fixed under $B_i^{(1)}$. But restricting to the fixed points and picking a basis of $\mathbb{C}^{n+1}$ that diagonalizes g, the basis vectors are in correspondence with boxes in the partition labeling the fixed point.
- Matrices $B_i^{(1)}$ act by moving the box in the i'th direction. We can thus see that the only one-dimensional subspaces of $\mathbb{C}^{n+1}$ preserved by the action of $B_i^{(1)}$ are those associated to the corners of the partition.

- The fixed points of $M(n+1,n)$ are thus labeled by a pair of partitions with $n+1$ and n boxes mutually related by an addition/removal of one box.
- For example, the fixed points of $M(3,2)$ for the D4-brane moduli are given by pairs



- The maps p and q project onto the first/second component and give a fixed point of $M(3)$ and $M(2)$ respectively.
- The weights of the added/removed box are respectively $2\epsilon_2, \epsilon_1, \epsilon_2, 2\epsilon_1$.