

Branes, Quivers and BPS algebras

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2.7. Recapitulation

- We have argued that the derived category of coherent sheaves form a good model of branes and their bound states:
 - We found coherent sheaves associated with a stack of branes supported on subvarieties inside \mathbb{C}^3 .
 - Non-reduced schemes have a physical interpretation in terms of turning on an expectation value for the Higgs field.
 - Complexes of sheaves can be interpreted as bound states of branes with a tachyonic field of non-trivial profile.
 - Quasi-isomorphisms then encode the processes of tachyon condensation.
- Morphisms $Hom^n(A, B)$ in the brane category correspond to massless string modes.
- $Hom^n(A, B)$ can be computed as morphisms $Hom(\tilde{A}, \tilde{B})$ between projective resolutions of our branes in the homotopy category (chain maps modulo chain homotopies).

2.8. Supersymmetric quantum mechanics

- We are now going to adapt the above tools to derive framed quivers with potential describing the low-energy dynamics of D0-branes bound to a fixed configuration of non-compactly supported branes.
- The low-energy dynamics of D0-branes is captured by a supersymmetric gauged quantum mechanics with potential.
- Such a quantum mechanics is specified by
 - a gauge group G specifying fields forming a vector multiplet,
 - a representation M of the group G specifying fields forming a chiral multiplet,
 - a holomorphic functions on M invariant under G called superpotential W .

See e.g. [\[Ohta-Sasai \(2014\)\]](#) for details.

- Today, we are now going to derive this data from calculations in the derived category of coherent sheaves, see e.g. [\[Sharpe \(2003\)](#), [Aspinwall-Katz \(2004\)](#), [Butson-MR \(in progress\)\]](#).

2.9. The gauge node

- The pair (G, M) coming from branes on a toric Calabi-Yau threefold can be encoded in terms of a framed quiver diagram.
- The gauge group G is going to be generally a product of $U(n_i)$ factors, each associated to a generator of the subcategory of compactly-supported branes.
- Since all the compactly supported branes in our \mathbb{C}^3 example are D0-branes, we have a single node with label n specifying the number of such D0-branes.
- Diagrammatically, we associate a circular node with each $U(n_i)$ factor and attach an integer n to it.

$$G = U(n_1) \times \dots \times U(n_m)$$

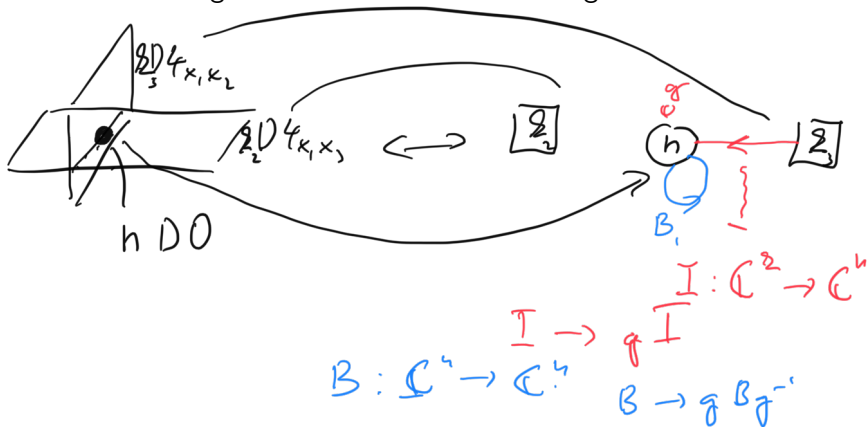
$$\textcircled{n_1} \quad \dots \quad \textcircled{n_m}$$

$$U(n) \xleftarrow{n \text{ D0}}$$

$$\textcircled{n}$$

2.10. The framing node

- We associate a square (framing) node to each elementary non-compactly supported brane and attach an integer k_j to it determining the number of branes in the given stack.



2.11. Arrows

- The representation M is encoded by arrows in the quiver diagram joining different nodes.
- Each arrow is in correspondence with a factor in $M = \bigoplus_{\alpha} M_{\alpha}$:
- Each factor M_{α} is associated with a map $\mathbb{C}^{n_i} \rightarrow \mathbb{C}^{n_j}$ with n_i being the integer attached to the tail node and n_j the integer attached to the terminal node of the arrow.
- A generator $g \in U(n_i)$ of the gauge group G acts on all M_{α} associated with arrows ending at the corresponding node by multiplication from the left and on all M_{α} associated to arrows starting at the corresponding node by multiplication by g^{-1} from the right.
- Physically, (G, M) determine fields of the quiver QM we want to construct. In turn, such fields should arise from massless strings stretched between our branes computed by $\text{Hom}(A, B)$. We are thus going to identify the arrows with generators of $\text{Hom}(A, B)$.

- In order to arrive at the desired quivers, we need to:
 - Restrict to morphisms $\text{Hom}^{\textcircled{1}}(A, B)$ of ghost-number one since only these contribute to physical modes.
 - Shifts complexes associated D0-branes by one. As explained above, construction of a bound states requires the degree of one of the two branes to be shifted. We thus need to introduce a shifts of complexes associated D0-brane:

$$D0: \quad \bar{G}^{-4} \rightarrow \bar{G}^{-3} \rightarrow \bar{G}^{-2} \rightarrow \bar{G}^{-1}$$



$$D0[1]: \quad G \rightarrow G^3 \rightarrow G^5 \rightarrow G$$

$$A \rightarrow \underline{D0[1]}$$

2.12. D0-D0 strings

- Let us now write down generators of $\text{Hom}^1(D0[1], D0[1])$. We have for example generator b_1 given by

$$\begin{array}{ccccccc}
 \mathcal{O} & \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} & \mathcal{O}^3 & \xrightarrow{\begin{pmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_3 & x_1 & 0 \end{pmatrix}} & \mathcal{O}^3 & \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} & \mathcal{O} \\
 \downarrow \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & & \downarrow (1, 0, 0) & & \\
 \mathcal{O} & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O}
 \end{array}$$

and analogously for b_2, b_3 .

- This leads to the quiver:

2.13. D0-D6 strings

$$\begin{array}{ccccccc} & & & & & & \mathcal{O} \\ & & & & & & \downarrow \\ & & & & & & 1 \\ & & & & & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O} \end{array}$$

2.14. D0-D4 strings

$$\begin{array}{ccccccc} & & & & \mathcal{O} & \longrightarrow & \mathcal{O} \\ & & & & \downarrow & & \downarrow \\ & & & & \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} & & 1 \\ \mathcal{O} & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O} \\ & & & & & & \\ \mathcal{O} & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow & & & & \\ 1 & & (0 \ 0 \ 1) & & & & \\ \mathcal{O} & \longrightarrow & \mathcal{O} & & & & \end{array}$$

2.15. D0-D2 strings

$$\begin{array}{ccccccc}
 & & \mathcal{O} & \longrightarrow & \mathcal{O}^2 & \longrightarrow & \mathcal{O} \\
 & & \downarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} & & \downarrow 1 \\
 \mathcal{O} & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O} \\
 \\
 \mathcal{O} & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O} \\
 \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} -1 & 0 & 0 \end{pmatrix} & & & & \\
 \mathcal{O} & \longrightarrow & \mathcal{O}^2 & \longrightarrow & \mathcal{O} & & \\
 \\
 \mathcal{O} & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O} \\
 \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} & & & & \\
 \mathcal{O} & \longrightarrow & \mathcal{O}^2 & \longrightarrow & \mathcal{O} & &
 \end{array}$$

2.16. String modes

- For completeness, let us also write down dimensions of all $\text{Hom}^n(A, B)$:

$\dim \text{Hom}^n$	$n = 0$	$n = 1$	$n = 2$	$n = 3$
D0[1]-D0[1]	1	3	3	1
D0[1]-D6 D6-D0[1]		1	1	
D0[1]-D4 D4-D0[1]		1	1	
D0[1]-D2 D2-D0[1]	1	2	1	
		1	2	1

2.17. Potential

- Let x_i for $i \in \text{arrows}$ be generators of $\text{Hom}^1(A, B)$ between various elementary branes in a given background. Any element in $\text{Hom}^1(A, B)$ can be then written as a linear combination

$$\Psi = \sum_{i_k \in \text{arrows}} X_k x_i$$

where $X_k : \mathbb{C}^{n_i} \rightarrow \mathbb{C}^{n_j}$ for n_i, n_j ranks associated with the tail and the head of the arrow i . In the string-field-theory literature, this linear combination is called the string field.

- The potential (as a function of X_k) is generally given by an A_∞ structure μ_m of the brane category together with a trace map (related to Serre duality) \int in terms of

$$W = \sum_{k=2}^{\infty} \frac{1}{k+1} \int \mu_2(\Psi, \mu_m(\Psi, \dots, \Psi))$$

- Strings can mutually join and split, leading to an associative product (often called the star product $\mu_2(\alpha_1, \alpha_2) = \alpha_1 \star \alpha_2$ in the string-field-theory literature)

$$\star : \text{Hom}^*(A_1, A_2) \otimes \text{Hom}^*(A_2, A_3) \rightarrow \text{Hom}^*(A, C)$$

- More generally, there also exist higher products

$$\underbrace{\text{Hom}^*(A_1, A_2) \otimes \cdots \otimes \text{Hom}^*(A_n, A_{n+1})}_n \rightarrow \text{Hom}^*(A_1, A_{n+1})$$

forming an A_∞ -structure.

- Luckily, these are trivial for \mathbb{C}^3 and the potential is simply

$$W = \int \Psi \star \Psi \star \Psi$$

with \star given by the composition of morphisms.

- Note also the symmetry of the first table above

$$\dim \operatorname{Hom}^n(D0, A) = \dim \operatorname{Hom}^{3-n}(A, D0)$$

- This is a consequence of the Serre duality stating that there exists a natural pairing

$$\operatorname{Hom}^n(D0, A) \times \operatorname{Hom}^{3-n}(A, D0) \rightarrow \mathbb{C}$$

- This pairing can be written as

$$\int \alpha \star \beta$$

where \int is known as a trace map.

2.18. Contribution of D0-D0 strings

2.19. Contribution of D0-D4 strings

2.20. Turning on Higgs field

- We would like to now comment on how to turn on the expectation value for the Higgs field on non-compact branes.
- Into the quiver, one can obviously include the modes of the non-dynamical fields coming from strings stretched between non-compact branes.

- This is going to lead to a modification of the potential:

- Turning on a constant value for such a Higgs field (that has to be nilpotent to preserve equivariance) leads to a modification of equations of motion.

2.21. Flavor symmetries

- Let us now look at $U(1)$ flavor symmetries for which we will introduce the Ω -background.
- Obviously, we can act by $GL(k)$ on the vector space associated to each framing node. Turning on the equivariance for its Cartan subgroup $U(1)^k \subset GL(k)$ plays an important role in understanding the framing by multiple branes and we will briefly comment on this point at the very end of our journey.
- Instead, note that the potential is invariant under

$$B_i \rightarrow e^{i\epsilon_i} B_i, \quad J \rightarrow e^{ia} J$$

for a being a linear combination of ϵ_i with integral coefficients depending on the choice of the framing brane if we restrict to the subtorus $U(1)^2 \subset U(1)^3$ given by $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$.

- This action on B_i can be traced back to the symmetry of the system that rotates the three coordinate planes in \mathbb{C}^3 .
- Let us write by $\mathbb{C}_{n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3}$ for integers n_i the representation of $U(1)^3$ given by

$$\mathbb{C}_{n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3} \rightarrow e^{i(n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3)} \mathbb{C}_{n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3}$$

- We can then lift the projective resolution of the D0-brane into the equivariant complex

$$\mathcal{O} \begin{pmatrix} -x_1 \\ x_2 \\ -x_3 \end{pmatrix} \rightarrow \mathcal{O} \times \mathbb{C}_{\epsilon_1} \oplus \mathbb{C}_{\epsilon_2} \oplus \mathbb{C}_{\epsilon_3} \xrightarrow{\begin{pmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & x & 0 \end{pmatrix}} \rightarrow$$

$$\mathcal{O} \otimes \mathbb{C}_{\epsilon_2+\epsilon_3} \oplus \mathbb{C}_{\epsilon_1+\epsilon_3} \oplus \mathbb{C}_{\epsilon_1+\epsilon_2} \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} \mathcal{O} \otimes \mathbb{C}_{\epsilon_1+\epsilon_2+\epsilon_3}$$

- Lifting everything into the equivariant map, we see that B_i must transform as $\mathbb{C}_{\epsilon_i} \otimes \mathbb{C}^{n^2}$

- Remember that the trace map \int was given in terms of a linear map $\text{Hom}^3(D0[1], D0[1]) \rightarrow \mathbb{C}$ sending a fixed generator to one. Lifting to an equivariant map, this generator is of weight $e^{i(\epsilon_1 + \epsilon_2 + \epsilon_3)}$. The condition $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ can be thus traced back to the requirement of the invariance of the trace map.
- Let me also mention a slightly different perspective. If we were to deal with D6-branes, we would identify $\text{Hom}^*(D6, D6) = H_{\bar{\partial}}^{*,0}(\mathcal{O}_X)$ with the trace map being the 6d holomorphic Chern-Simons functional

$$\int_X \alpha \wedge \Omega$$

where Ω is the Calabi-Yau volume form. In our case, $\Omega = dx_1 \wedge dx_2 \wedge dx_3$ and we can see that its invariance requires $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$.