Geometry of $\mathcal{N}=2$ theories

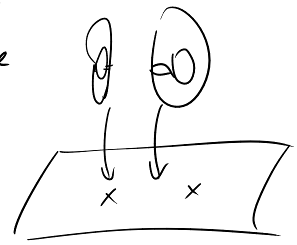
$\mathcal{N}=2$ theories

e.g. $\boxed{T(G,R)}$ "Lagrangian field theory"

$T(\mathfrak{g}, C)$ "theory of class S"

$T(g_s, Y)$ "geometric engineering"

complex integrable systems



hyperkähler metrics

quantization

BPS states (enumerative invariants)

Donaldson invariants

Suppose given a smooth compact 4-manifold X.

Idea: to study top. invts of X, equip X w/ Riem metric g; study diff. eq. on (X,g)

Baby example: $\Delta : \Omega^k(X) \to \Omega^k(X)$ $\qquad \Delta\alpha = dd^*\alpha + d^*d\alpha$

consider $\mathcal{H}^k(X,g) = \{\alpha \in \Omega^k(X) : \Delta\alpha = 0\}$

linear PDE on (X,g)

Riemannian Hodge thm: $b_k := \underset{\in \mathbb{Z}}{\dim_{\mathbb{R}} \mathcal{H}^k(X,g)}$ is k-th Betti number of X
— in ptic, independent of g — top. invariant of X!

Non-baby example (Donaldson): let $G = SU(2)$

consider $\mathcal{M}(X,g) = \{G\text{-bundles over } X \text{ with connection } \nabla : F_+^{\nabla} = 0\}$

nonlinear PDE on (X,g) (b/c $SU(2)$ not abelian)

$F_+ = \frac{1}{2}(F + \star F)$

Not a linear space — has components of various dimensions

$$\mathcal{M} = \bigcup_{k \geq 0} \mathcal{M}_k, \qquad \dim \mathcal{M}_k = 8k - 3(1 - b_1(X) + b_2^+(X))$$

$\mathcal{M}_k$ has natural orientation, natural closed differential forms $\tau_\alpha \in \Omega^*(\mathcal{M})$ labeled by classes $\alpha \in H_*(X)$ $\qquad \deg \tau_\alpha = 4 - \deg\alpha$

Then consider integrals
$$\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_\ell} \rangle_{Don} = \int_{\mathcal{M}} \tau_{\alpha_1} \wedge \tau_{\alpha_2} \wedge \cdots \wedge \tau_{\alpha_\ell}$$

These are topological invariants of X (as long as $b_2^+(X) > 1$).

Difficult to work with — e.g. because $\mathcal{M}$ is typically not compact

Witten 1988: (Donaldson invariants)

$$\cap$$

$$(\text{4-dim } \mathcal{N}=2 \text{ super YM theory}) \quad G=SU(2),\ R=\phi$$

$$\cap$$

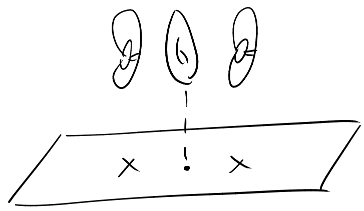
$$(\text{4-dimensional } \mathcal{N}=2 \text{ SUSY QFT})$$

$$\cap$$

$$(\text{4-dimensional QFT})$$

Seiberg-Witten 1994: "solution" of the low energy dynamics of 4d $\mathcal{N}=2$ SYM with $\underline{G=SU(2)}$, $R=\phi$

(wrote down the EFT at low energies)



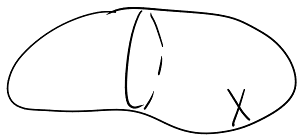
Four-dimensional QFT

Roughly: a 4d QFT T is a machine which assigns objects $Z_T(X)$ to manifolds X of dim ≤ 4 —

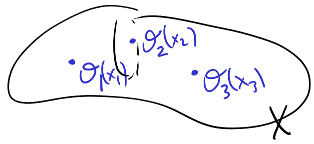
X maybe equipped with e.g. spin structure, Riem metric, framing, H-connection ("global symmetry gp" H)

defects supported on submanifolds $Y \subset X$

e.g. X = closed 4-manifold — $Z_T(X) \in \mathbb{C}$

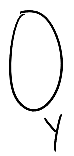


$$Z_T(X) = \langle\ \rangle_T \in \mathbb{C}$$



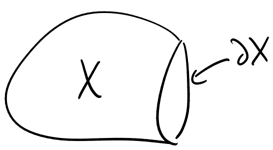
$$Z_T(X \text{ w/ insertions}) = \langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3) \rangle_T \in \mathbb{C}$$

Y: closed 3-manifold



$$Z_T(Y) \in \mathrm{Vect}$$

X = 4-mfd with boundary



$$Z_T(X) \in Z_T(\partial X)$$

obeying gluing rules ...

Very rich structure!
We'll use only a small part.

SYM Given: • compact Lie gp G
• f-d $\mathbb{C}$ representation R of G (suff. small)

- $\tau_{UV} \in \mathbb{C}$ $\quad (*)$

$\exists$ $\mathcal{N}=2$ SUSY_{4d} QFT $\quad T(G, R, \tau_{UV})$

Spacetime X carries:
- Riem. metric
- spin structure
- $H = [\underbrace{SU(2)}]$ - connection

"R-symmetry" (take trivial for a while)

How to construct $T(G, R, \tau_{UV})$?
By quantization of a classical field theory —
given X we consider a field space $\mathcal{F} = \mathcal{F}(X)$ and an "action" $S : \mathcal{F} \to \mathbb{C}$
then try to define
$$Z_T(X) = \langle \rangle = \int_{\mathcal{F}} d\mu \; e^{-S} \in \mathbb{C}$$

given a local function $\mathcal{O}(x) : \mathcal{F} \to \mathbb{C}$
we can also try to define
$$Z_T(X \text{ with } \mathcal{O}(x) \text{ inserted}) = \langle \mathcal{O}(x) \rangle = \int_{\mathcal{F}} d\mu \; \mathcal{O}(x) \, e^{-S} \in \mathbb{C}$$

and similarly $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$

Two caveats:
- $\mathcal{F}$ is most conveniently described by giving over-large collection of objects and equivalences between them — i.e. $\mathcal{F}$ is groupoid, not set
- $\mathcal{F}$ is actually a superspace, so the "function" $S : \mathcal{F} \to \mathbb{C}$ understood in sense of supergeometry.

In case $R = \emptyset$:
- Bosonic part of set of objects of $\mathcal{F}$ is
$$\underline{\mathcal{F}_{bos} = \{ (E, \varphi, D) \}}$$
where:
  - E is princ. G-bundle $\to X$ w/ conn ∇
  - φ is a section of $E \times_G \mathfrak{g}_{\mathbb{C}}$ $\quad$ i.e. $(\mathrm{ad}\, E)_{\mathbb{C}}$
  - D is a section of $E \times_G (\mathfrak{g}_{\mathbb{C}} \otimes \mathbb{C}^3)$ ("auxiliary field")
- Arrows $(E, \varphi, D) \to (E', \varphi', D')$ in $\mathcal{F}$ are princ. bundle maps $g : E \to E'$
$g^* \varphi' = \varphi$, $g^* D' = D$, $g^* \nabla' = \nabla$
- The action is
$$S|_{\mathcal{F}_{bos}}(E, \varphi, D) = \int_X d\mathrm{vol}\; \tau_{UV} \| F_+^{\nabla} \|^2 + \bar{\tau}_{UV} \| F_-^{\nabla} \|^2 + (\mathrm{Im}\,\tau) \left(\| \nabla \varphi \|^2 + \| [\varphi, \varphi^\dagger] \|^2 + \| D \|^2 \right)$$

S has a lot of symmetry.

Manifestly invariant under $\widetilde{\mathrm{Isom}}(X) \circlearrowright F$

e.g. if $X = \mathbb{R}^4$, S is invariant under $\widetilde{\mathrm{Isom}}(\mathbb{R}^4) = I\,\mathrm{Spin}(4)$

In fact S is inv't under a super Lie algebra $A_{\mathcal{N}=2,\, d=4}$ extending $i\mathrm{spin}(4)$

"$\mathcal{N}=2$ supersymmetry algebra"

To write it:

$\mathrm{Spin}(4) \simeq \underline{SU(2)} \times \underline{SU(2)}$ has 2 inequivalent spin representations $S^\pm$ (complex, 2-dim)

w/ inv't skew pairing $\eta: S^\pm \otimes S^\pm \to \mathbb{C}$

$\Gamma: S^+ \otimes S^- \xrightarrow{\sim} V$ $\quad$ V vector rep of $\mathrm{Spin}(4)$

As rep. of $I\mathrm{Spin}(4)$ it's

$$A_{\mathcal{N}=2,d=4} = \overbrace{\left(i\mathrm{spin}(4) \oplus \mathbb{C}^2 \right)}^{\text{even part}} \oplus \overbrace{\left(S^+ \oplus S^+ \oplus S^- \oplus S^- \right)}^{\text{odd part}}$$

$\leftarrow$ 2 each of $S^\pm$ "$\mathcal{N}=2$"

rotations, translation P_μ, $\mu = 1,\dots,4$

$Z, \bar{Z}$ central

Q^1_α $\quad Q^2_\alpha$ $\quad \bar{Q}^1_{\dot\alpha}$ $\quad \bar{Q}^2_{\dot\alpha}$

$\alpha=1,2$ $\quad \alpha=1,2$ $\quad \dot\alpha=1,2$ $\quad \dot\alpha=1,2$

$$\underline{\underline{\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = \delta^{IJ}\Gamma^\mu_{\alpha\dot\beta} P_\mu}} \qquad \{Q^I_\alpha, Q^J_\beta\} = \epsilon^{IJ}\eta_{\alpha\beta} Z$$

In fact A is rep of $I\mathrm{Spin}(4) \times H$ $\leftarrow$ $H = SU(2)$ "R-symmetry"

then odd part is $\quad S^+ \otimes \underset{\underset{H}{\circlearrowleft}}{\mathbb{C}^2} \oplus S^- \otimes \underset{\underset{H}{\circlearrowleft}}{\mathbb{C}^2}$

Twisted description

New action of $I\mathrm{Spin}(4)$ on A:

$$I\mathrm{Spin}(4) \xrightarrow[(1,\rho)]{} \underbrace{I\mathrm{Spin}(4)}\times\underbrace{H} \longrightarrow \mathrm{Aut}(A)$$

$$\begin{tikzcd} \rho: I\mathrm{Spin}(4) \arrow[r] \arrow[d] & H = SU(2) \\ \mathrm{Spin}(4) = SU(2)\times SU(2) \arrow[ur, "(g_+, g_-) \mapsto g_+"'] & \end{tikzcd}$$

As rep of $ISpin(4)$:

$$A_{N=2,d=4} = (\underbrace{Lie(ISpin(4)) \oplus \mathbb{C}^2}_{\text{even}}) \oplus (\overset{4}{\underset{Q_\mu}{V}} \oplus \overset{1}{\underset{Q}{\mathbb{C}}} \oplus \overset{3}{\underset{Q_{(\alpha\beta)}}{\Lambda^{2,+}V}})$$

with brackets

$$\{Q,Q\} = 0, \quad \{Q,Q_\mu\} = P_\mu, \quad \ldots$$

Twisted theory

Given Riem 4-mfd X w/ spin str

introduce $\underset{SU(2)}{H}$-bundle $\to X$ which is the projection of Levi-Civita conn. along P

$$Spin(4) = SU(2) \times SU(2) \longrightarrow SU(2) = H$$

Then, can define $\mathcal{F} = \mathcal{F}(X,P)$ essentially as before, $S : \mathcal{F} \to \mathbb{C}$,

and then have $\underset{Lie(\mathbb{R}^{0|1})}{\underline{Q}}$ acting on $\mathcal{F}$ preserving S.

Want to compute $Z = \langle \, \rangle$ "=" $\int_{\mathcal{F}} d\mu \; e^{-S}$

In general QFT, hard.

But the Q symmetry helps us here: SUSY localization.

Basic philosophy: (1) for any local operator $\mathcal{O}$,

$$\langle Q\mathcal{O} \rangle = 0 \qquad \qquad QS = 0, \quad Q(e^{-S}) = 0$$

(Path integral reasoning: $\langle Q\mathcal{O} \rangle = \int_{\mathcal{F}} (Q\mathcal{O})\, e^{-S} = \int_{\mathcal{F}} Q(\mathcal{O} e^{-S}) = 0$).

(2) Pick some $\Psi : \mathcal{F} \to S$

and deform the theory by $S \to S + tQ\Psi$

Then $\partial_t Z = \partial_t \int_{\mathcal{F}} e^{-(S+tQ\Psi)} = -\int_{\mathcal{F}} (Q\Psi)\, e^{-(S+tQ\Psi)} = -\langle Q\Psi \rangle_t = 0$

So Z is independent of t.

But we can choose Ψ so that $Q\Psi > 0$ everywhere away from Q-fixed locus in $\mathcal{F}$.

$\rightsquigarrow$ reduce Z to an integral over Q-fixed locus $\subset \mathcal{F}$

In the case $T = N=2$ SYM with $G = SU(2)$

this Q-fixed locus is = instanton moduli space $\mathcal{M}$

$\rightsquigarrow Z = \langle \, \rangle = \langle \, \rangle_{\text{Donaldson}}$

What about $\langle \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_n} \rangle_{\text{Donaldson}}$?

V = local operators in theory on $\mathbb{R}^4$

$\circlearrowright$

$A_{\mathcal{N}=2, d=4}$

$Q^2 = 0$

Consider $\mathcal{R} = \dfrac{\ker Q}{\operatorname{im} Q}$

If $\underline{\underline{Q\mathcal{O}=0}}$ call $\mathcal{O}$ "Q-closed"

if $\mathcal{O} = Q\Psi$ —— "Q-exact"

$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$ have nice properties when all $\mathcal{O}_i$ are Q-closed:

- $\langle \cdots \rangle$ depends only on the Q-coh. classes of the $\mathcal{O}_i$
- $\langle \cdots \rangle$ is independent of the insertion points x_i
- $\langle \cdots \rangle$ is indep. of the metric on X (if $b_2^+(X) > 1$)

These follow from

$$0 = \langle Q(\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)) \rangle = \sum_{i=1}^{n} \langle \mathcal{O}_1(x_1) \cdots Q\mathcal{O}_i(x_i) \cdots \mathcal{O}_n(x_n) \rangle$$

The $\mathcal{O}$ with $\underline{\underline{Q\mathcal{O}=0}}$ also have $\underline{\text{descendants}}$: define

$$\mathcal{O}^{(1)} = (Q_\mu \mathcal{O})\, dx^\mu \qquad\qquad Q\mathcal{O}^{(k)} = d\mathcal{O}^{(k-1)}$$

$$\mathcal{O}^{(2)} = \tfrac{1}{2}(Q_\mu Q_\nu \mathcal{O})\, dx^\mu \wedge dx^\nu \qquad\qquad (\text{using } \{Q, Q_\mu\} = P_\mu)$$

$$\mathcal{O}^{(3)} = \text{— —}$$

$$\mathcal{O}^{(4)} = \text{— —}$$

Then define for $\underset{k\text{-chain}}{\Gamma \subset \mathbb{R}^4}$ (or $\Gamma \subset X$) $\qquad \mathcal{O}(\Gamma) = \oint_\Gamma \mathcal{O}^{(k)}$

This has $Q\mathcal{O}(\Gamma) = \mathcal{O}(\partial\Gamma)$

so if Γ is closed, $Q\mathcal{O}(\Gamma) = 0$

Then $\langle \mathcal{O}_1(\Gamma_1) \cdots \mathcal{O}_n(\Gamma_n) \rangle$ has good properties similar to those of $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$.

Formally: $\langle \mathcal{O}_1(\Gamma_1) \cdots \mathcal{O}_n(\Gamma_n) \rangle = \langle \mathcal{O}_1(\Gamma_1) \cdots \mathcal{O}_n(\Gamma_n) \rangle_{\text{Donaldson}}$.

Now, go back to the theory on $X = \mathbb{R}^4$.

$$R = \frac{\ker Q}{\operatorname{im} Q}.$$

Consider all possible values of 1-point functions $\langle \mathcal{O} \rangle$ for $\mathcal{O} \in R$

R is a ring: if $\mathcal{O}_1, \mathcal{O}_2 \in R$ $\quad \lim_{x_1 \to x_2} [\mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)]$

$$\lim_{x_1 \to x_2} [\mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)] = (\mathcal{O}_1 \cdot \mathcal{O}_2)(x_2)$$

$x_1 \to x_2$

$$\langle \mathcal{O}_1 \cdot \mathcal{O}_2 \rangle = \langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \rangle = \lim_{\|x_1 - x_2\| \to \infty} \langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle$$

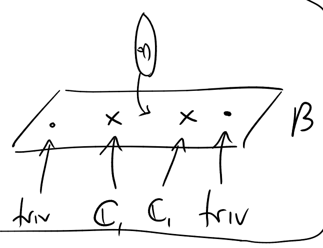
So $\{$possible values of 1-pt functions$\} = \operatorname{Spec} R$

"Coulomb branch"

$\mathcal{N}=2$ SYM theory
$G = SU(2)$
$R = $ triv

low energy behavior $\rightsquigarrow$

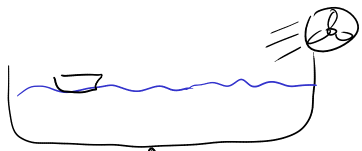
$\mathcal{N}=2$ SYM theory
$G = U(1)$
$R = $ triv or $\mathbb{C}_1$



Donaldson theory

Seiberg-Witten theory

Analogy:



$\circ_\circ^\circ$ $\sim 10^{27}$ particles
"high energy"/"UV" description

$\rightsquigarrow$

Navier-Stokes equations
velocity, pressure, density, viscosity

the hard and important step!

Last time: consider local operators in $\mathcal{N}=2$ theory ($d=4$) $\qquad R = \dfrac{\ker Q}{\operatorname{im} Q} \qquad \mathcal{B} = \operatorname{Spec} R$

In $\mathcal{N}=2$ SYM w/ gp G

R = G-invariant polynomials P on $\mathfrak{g}$ $\quad (\mathcal{O}_P = P(\varphi)) \quad \varphi \in \mathfrak{g}_\mathbb{C}$

e.g. if $G = SU(2)$

then $R = \mathbb{C}[\mathcal{O}_P]$ $\quad P(X) = \mathrm{Tr}\, X^2$ inv't poly on $\mathfrak{g} = su(2)$

if $G = SU(N)$

$R = \mathbb{C}[\mathcal{O}_{P_2}, \mathcal{O}_{P_3}, \dots, \mathcal{O}_{P_N}]$ $\quad P_k(X) = \mathrm{Tr}\, X^k$

So $\mathcal{B}$ is an affine space of dim $= N-1$

$(\mathcal{B} \simeq \mathbb{C}^{N-1})$

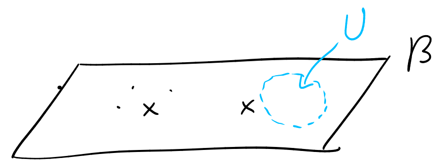
when $N=2$, $\mathcal{B} \simeq \mathbb{C} \ni u$ $\qquad u = \langle \mathrm{Tr}\, \varphi^2 \rangle$

Now suppose R = trivial.

Then, S-W say: consider family of curves $\Sigma_u = \{ y^2 = x^{-2}(x + 2u + x^{-1}) \} \subset \mathbb{C}^2_{x,y}$

(thrice-punctured tori, smooth except for $u = \pm 1$)

equipped with 1-form $\lambda = y\, dx$



let $\Gamma_u = H_1(\Sigma_u, \mathbb{Z}) \simeq \mathbb{Z}^2$ for $u \neq \pm 1$

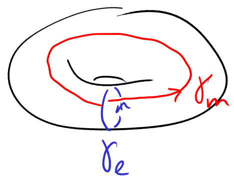
(a superlattice of)

Γ gives local system of lattices over $\mathcal{B}' = \mathcal{B} \setminus \{1, -1\}$.

and for $\gamma \in \Gamma_u$ define $Z_\gamma = \oint_\gamma \lambda$.

Choose a s.c. patch $U \subset \mathcal{B}'$

and basis $\{\gamma_e, \gamma_m\}$ for Γ_u, $u \in U$.



Then, let $a := Z_{\gamma_e}$ and $a_D := Z_{\gamma_m}$ $\qquad a : U \to \mathbb{C}$, $a_D : U \to \mathbb{C}$ hol.

$$\left[\tau = \frac{i}{g^2} + \frac{\theta}{2\pi} \right]$$

a gives a local coordinate

so can write $a_D(a)$ and define $\tau(a) := \dfrac{da_D}{da}$.

S-W say: Correlation functions $\langle \dots \rangle_u$ of the theory $T(SU(2), \mathrm{triv})$ in $X = \mathbb{R}^4$

depend on a parameter $u \in \mathcal{B}$.

For $u \in U$, they are approximately described by theory $T(U(1), \mathrm{triv.}, \tau(a_0))$

— effective action in terms of fields (E, a, D) $\qquad E$ $U(1)$-bundle with connection

$a : \mathbb{R}^4 \to \mathcal{B}$ (expand around a_0)

$$S\Big|_{F_{bos}} = \int_{\mathbb{R}^4} d\mathrm{vol}\ \tau(a)\,\|F_+^\nabla\|^2 + \bar\tau(\bar a)\,\|F_-^\nabla\|^2 + \underbrace{(\mathrm{Im}\,\tau(a))}\left(\|da\|^2 + \|D\|^2\right)$$

special Kähler metric on $\mathcal{B}$

$[(\mathrm{Im}\,\tau(a))\,|da|^2]$

$$\left(= \frac{1}{g^2}\int \underset{F_{\mu\nu}F^{\mu\nu}}{F\wedge *F} + \frac{i\theta}{2\pi}\int F\wedge F + \cdots \right)$$

i.e. there is a map between the operators $\mathcal{O}$ of $T(SU(2),\text{triv})$ and $\mathcal{O}^{IR}$ of $T(U(1),\text{triv},\tau(a_0))$

s.t. $\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_{T(SU(2),u)} \approx \langle \mathcal{O}_1^{IR}(x_1)\cdots\mathcal{O}_n^{IR}(x_n)\rangle_{T(U(1),a_0)}$

$\approx$ means up to corrections which decay exp. in $\|x_i - x_j\|$.

This statement is particularly powerful if $\langle\cdots\rangle$ is indep of the x_i's

(e.g. if all $\mathcal{O}_i \in R$, or for corr. of descendants on compact X in the twisted description).

Duality. Our description depended on choice of a basis for $\Gamma \simeq \mathbb{Z}^2$.

Different choices make a diff:

changing basis by $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})$

changes $\tau \longrightarrow \tau' = \dfrac{a\tau+b}{c\tau+d}$.

How can the same theory be described both by $T(U(1),\tau)$ and $T(U(1),\tau')$?

Answer: those two theories are actually equivalent!

(electric-magnetic duality of $U(1)$ gauge theory (also in $N=2$ SUSY $U(1)$ theory))

Concretely: think of Γ as lattice of electromagnetic charges carried by particles in the effective theory

choosing basis $\{\gamma_e, \gamma_m\}$ corr. to choosing which charge is "electric" + which is "magnetic".

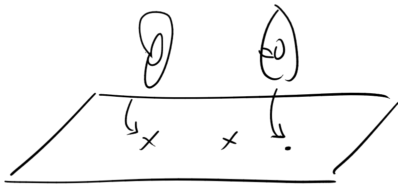
Complex integrable system

Let $\mathcal{M}_u = \mathrm{Hom}(\Gamma_u, U(1))$. Compact 2-torus, loc. coords $(\theta_e, \theta_m) \in U(1)^2$

The $\mathcal{M}_u$ fit into a family $\mathcal{M}$, naturally $\mathbb{C}$ int'ble system:

$\omega = da \wedge d\theta_m - da_D \wedge d\theta_e \in \Omega^2_{\mathbb{C}}(\mathcal{M})$

determines hol. sympl. str. on $\mathcal{M}$.

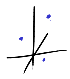


$\mathcal{M}$ extends over full $\mathcal{B}$.

(= periodic Toda lattice for $G = SU(2)$)

$\mathcal{N}=2$ SUSY $d=4$ QFT

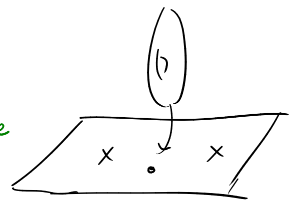
ex. $T(SU(2), \text{triv})$



$\rightsquigarrow$

$\mathbb{C}$ int'ble system

ex. periodic Toda lattice for $G = SU(2)$



$\mathcal{M}$

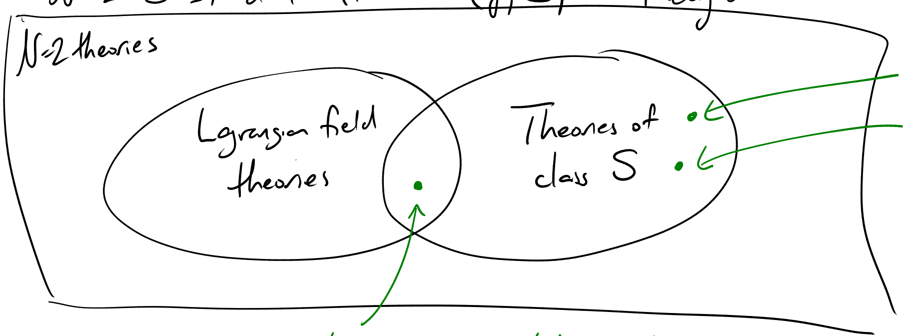
$\downarrow$

$\mathcal{B}$

(Coul. branch)

Given the data ($\mathfrak{g}$ = Lie alg of ADE type,
C = compact Riemann surface w/ punctures + puncture data)

there is an $\mathcal{N}=2$ SUSY $d=4$ QFT $S(\mathfrak{g}, C)$ "theory of class S"



$S(SU(3), \mathbb{CP}^1$ w/ 3 regular punctures)

$S(SU(3), C$ of genus 2)

$T(SU(2), \text{triv}) = S(A_1, \mathbb{CP}^1$ w/ 2 irregular punctures)

For the theory $S(\mathfrak{g}, C)$ the int. sys. $\mathcal{M}$ is the Hitchin system
i.e. moduli space of $\mathfrak{g}$-Higgs bundles over C.

$$= \left\{ (E, \varphi) : \begin{array}{l} E \text{ principal } G_{\mathbb{C}}\text{-bundle} \to C \\ \varphi \text{ hol. section of ad } \mathfrak{g}_{\mathbb{C}} \end{array} \right\}$$

The function $\tau(u)$ on $\mathcal{B}$ giving $\mathbb{C}$ str. of the torus fibers $\mathcal{M}_u$ gives low energy action of the $\mathcal{N}=2$ theory.

τ_{IJ}

How does the full space $\mathcal{M}$ come to life in the physics?

Consider the 4d theory "reduced on S^1"

i.e. define a 3d theory by $Z_{3d}(X) = Z_{4d}(X \times S^1)$.

Long-distance description of Z_{3d}:

by an IR QFT given by an effective action $S : \mathcal{F} \to \mathbb{R}$

where $\mathcal{F}_{bos}$ is space of maps $\varphi : X \to \mathcal{M}$

$$S|_{\mathcal{F}_{bos}} = \int \|d\varphi\|^2$$

requires a Riem. metric on $\mathcal{M}$

ie: this compactification produces a Riem. metric on $\mathcal{M}$, which is hyperkähler

Def A Riem metric $(\mathcal{M},g)$ is HK if $\mathcal{M}$ admits complex structures $I, J, K \in \mathrm{End}(T\mathcal{M})$ obeying $IJ=K,\ JK=I,\ KI=J$

s.t. g is Kähler wrt I, J, K

Rk (M,g) hyperkähler $\Longrightarrow$ g Ricci-flat

Fact If $(\mathcal{M},g)$ HK then for $(a,b,c)\in S^2$ $I_{(a,b,c)} = aI + bJ + cK$

is also a $\mathbb{C}$ str on $(\mathcal{M},g)$

and g is Kähler for it

It's a good idea to think of this S^2 as $\mathbb{CP}^1 \ni \zeta$

e.g. can build "twistor space" $Z(\mathcal{M})$ complex, $\pi^{-1}(\zeta) \simeq (\mathcal{M}, I_\zeta)$ as $\mathbb{C}$ mfd

$$Z(\mathcal{M}) \xrightarrow{\ \pi\ } \mathbb{CP}^1$$

(M,g) Kähler wrt $I, J, K \rightsquigarrow$ 3 symplectic forms $\omega_I, \omega_J, \omega_K$

e.g. in $\mathbb{C}$ str I, have Kähler form ω_I

hol. symplectic form $\varpi_I = \omega_J + i\omega_K$

in $\mathbb{C}$ str I_ζ, hol. sym. form $\varpi(\zeta) = \frac{\omega_+}{\zeta} + \omega_I + \omega_- \zeta$ $\qquad \omega_\pm = \omega_J \pm i\omega_K$

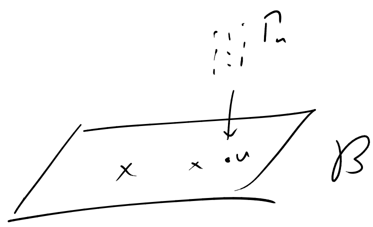
Determining this $\varpi(\zeta) \rightsquigarrow$ computing the HK metric.

Given $(\mathcal{M}, \varpi_I = \omega_J + i\omega_K)$ how to produce the whole HK structure?

Claim: we can do it using one more piece of data —

a function $\Omega : \Gamma \to \mathbb{Z}$

piecewise constant, obeying Kontsevich-Soibelman WCF



$$\varpi(\zeta) = \sum_{i=1}^{\mathrm{rk}\,\Gamma} \frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}\, \epsilon_{ij}$$

Conj There exist holomorphic Darboux coordinates

$$\mathcal{X}_\gamma : \mathcal{M} \times \mathbb{C}^\times \to \mathbb{C}^\times \quad (\gamma \in \Gamma)$$

$\zeta \in \mathbb{C}^\times$

γ_i basis of Γ

$\epsilon_{ij} = \langle \gamma_i, \gamma_j \rangle$

such that

ϕ_γ coords on torus fiber $=0$ at $\theta_\gamma = 0$

1. $\mathcal{X}_\gamma(\zeta) \sim \exp\left(\frac{Z_\gamma}{\zeta} + i\theta_\gamma + \delta_\gamma\right)$ as $\zeta \to 0$
2. $\mathcal{X}_\gamma(\zeta) = \overline{\mathcal{X}_{-\gamma}\left(-\frac{1}{\zeta}\right)}$
3. $\mathcal{X}_\gamma(\zeta)$ is piecewise hol. in ζ

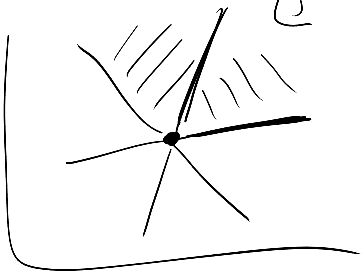
with jump of form

$$\mathcal{X}_\gamma \longrightarrow \mathcal{X}_\gamma (1 - \mathcal{X}_\mu)^{\boxed{\Omega(\mu)} \langle \gamma, \mu \rangle}$$

$(\mathbb{C}^\times)^{\mathrm{rk}\,\Gamma} \circlearrowleft$

at the ray $\ell_\mu := \{ \zeta \in \mathbb{C} \mid \arg \zeta = \arg(-Z_\mu) \}$, $\mu \in \Gamma$

[Dubrovin]
[Cecotti-Vafa]



One can try to find such $\mathcal{X}_\gamma$ by solving an integral equation, eg numerically

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Why? Physics of $\mathcal{X}_\gamma$:

$\mathcal{M}$ is Coulomb branch of a 3d theory (cf BFN: "K-theory version")

$$\mathcal{X}_\gamma(\zeta) = \langle \mathcal{O}_\gamma(\zeta) \rangle$$

$\mathcal{O}_\gamma(\zeta)$ is a local op. obtained from taking a line op. of 4d $\mathcal{N}=2$ theory wrapped around S^1

line ops in image of UV-IR map for line ops

$$\oint_P \exp\left(\frac{\varphi}{\zeta} + iA + \varphi^\dagger \zeta\right) \quad \zeta \in \mathbb{C}^\times$$

$\gamma \in \Gamma$ corresponds to line $L_\gamma^{(\zeta)}$: Wilson-'tHooft line w/ electromagnetic charge γ.

Physics of $\Omega(\gamma)$: $\Omega(\gamma) \in \mathbb{Z}$ is BPS index counting BPS states of charge γ Hilbert space of 4d theory.
