Counting social interactions for discrete subsets of the plane

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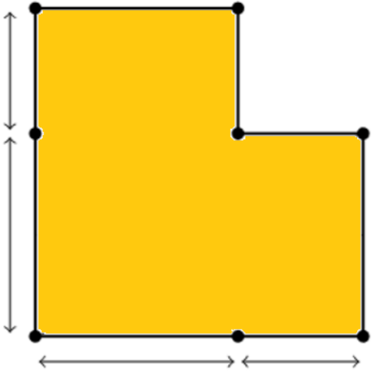
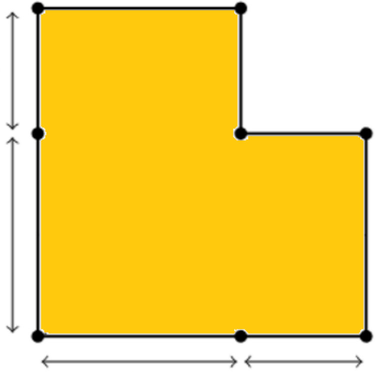
Overview

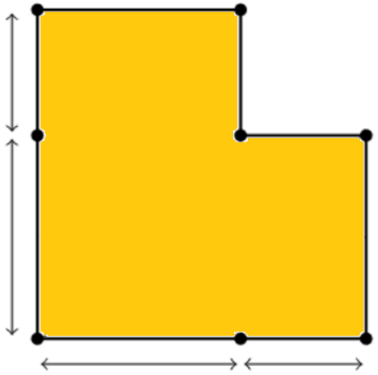
1. The Golden L and holonomy vector population density
2. Counting closed geodesics via Γ-orbits
3. Expected populations on n-street
4. Few nearby neighbors
5. BREAK
6. Higher moments of the Siegel–Veech transform
7. Proof ideas: Orbit decomposition and counting orbits

The Golden L



Holonomy Vectors on the Golden L





Veech '98

Set of closed geodesics are finite union of H_5 orbits.

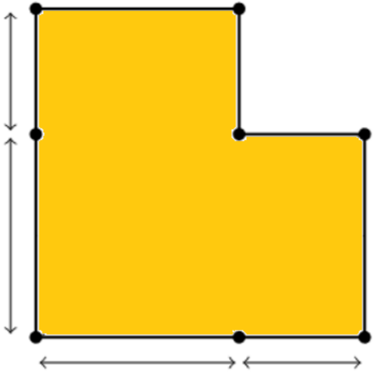
$$\Lambda_5 = H_5 \cdot e_1 \sqcup H_5 \cdot \overline{u} e_1$$

$$u = \frac{1+\sqrt{5}}{2}$$

$$H_q = \left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2\cos\left(\frac{\pi}{q}\right) \\ 0 & 1 \end{bmatrix} \right\rangle$$

Looking at one orbit

$$V = H_5 \cdot e_1 \qquad H_q = \left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2\cos\left(\frac{\pi}{q}\right) \\ 0 & 1 \end{bmatrix} \right\rangle$$



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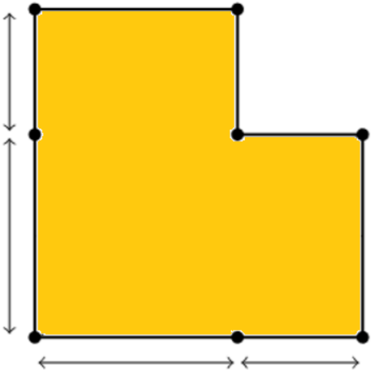
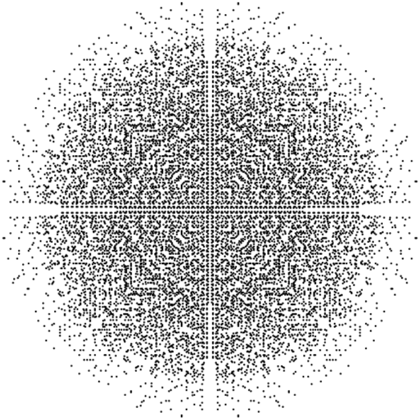
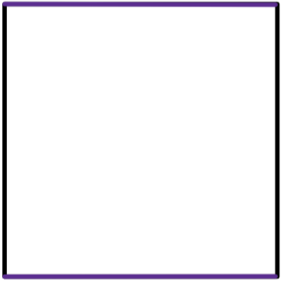
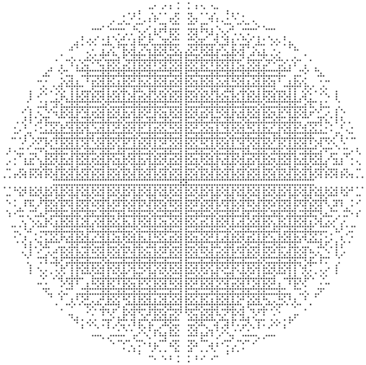




Our friend the Torus

$$V = H_3 \cdot e_1 \qquad H_q = \left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2\cos\left(\frac{\pi}{q}\right) \\ 0 & 1 \end{bmatrix} \right\rangle$$

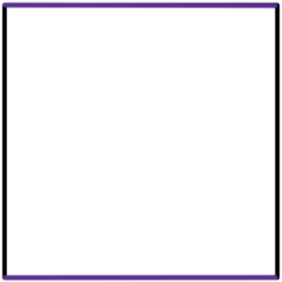


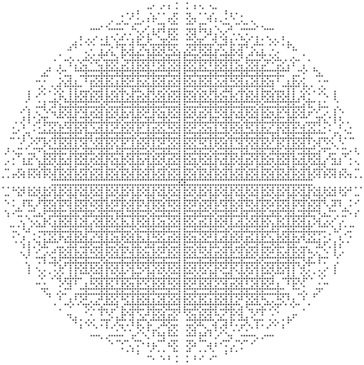


Population Density on the torus

Assuming Riemann Hypothesis (Wu, 2002)

$$\#\{V \cap B(0, R)\} = \frac{6}{\pi^2}(\pi R^2) + O(R^{\frac{221}{304}+\epsilon})$$

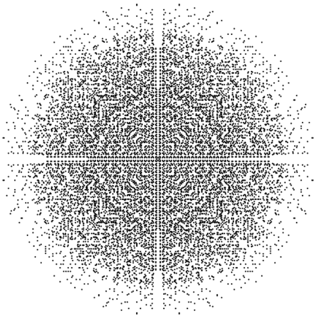




Population density on the Golden L

Theorem (BNRW 2019)

$$\#\{V \cap B(0,R)\} = \frac{10}{3\pi^2} \cdot \pi R^2 + O(R^{\frac{4}{3}})$$



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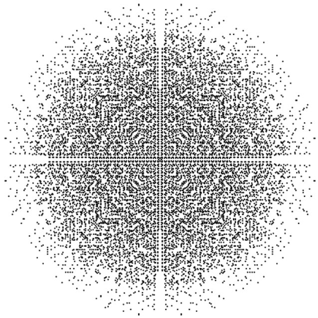

Theorem (Burrin-F., Coming soon!)

Ω bounded Jordan measurable domain

$$\mathbb{E}(\#\{V \cap R \cdot \Omega\}) = \frac{10}{3\pi^2} \cdot |\Omega| R^2 + O(R^c)$$

where $c = \max\{\frac{4}{3}, 2s_1\}$.

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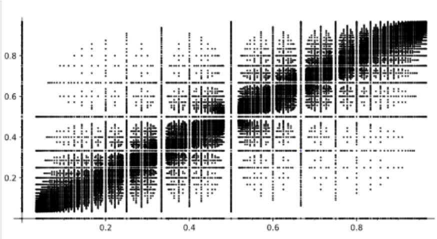
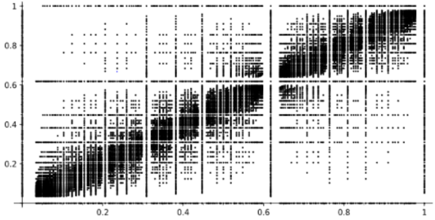
Big proof idea: Count pairs of vectors in V!





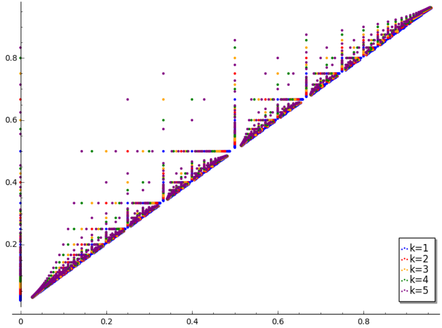
Given $v, w \in V \cap B(0, 30)$ with $|v \wedge w| < 30$ plot $\left(\frac{v_2}{v_1}, \frac{w_2}{w_1}\right)$

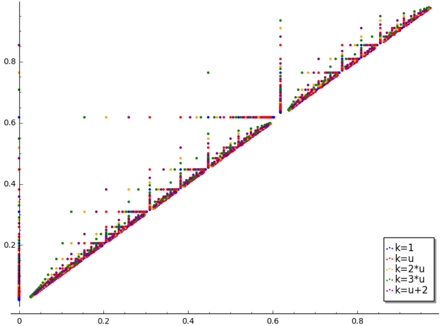
Population density on nth street

Counting Pairs by determinant (F. 2019)

$$\mathbb{E}(\{v, w \in V \cap B(0, R) : |v \wedge w| = n\}) \sim \frac{10}{3\pi^2} \cdot \frac{\pi^2}{n} \cdot \varphi(n) \cdot R^2$$

Population density on *n*th street





Density of nearby neighbors

Corollary to F.2019, Coming soon!

For all $\delta > 0$, there exists $\epsilon > 0$ so that

$$\limsup_{R \to \infty} \frac{\#\{v \in V \cap B(0,R) : \exists\, w \in V \cap B(v,\epsilon)\}}{R^2} < \delta.$$

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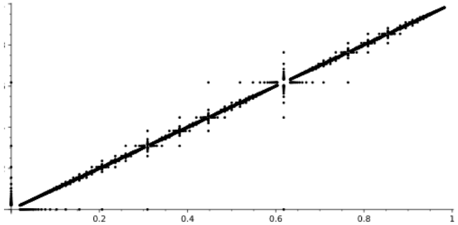

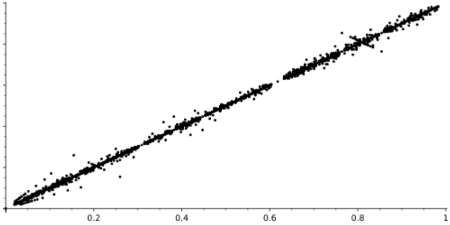

$$v, w \in V \cap B(0, 50)$$

$$|v \wedge w| = 1 \quad || \quad w \in B(v, 1/2)$$

Break

Siegel–Veech Integral Formula

$\Gamma < SL(2,\mathbb{R})$ non-uniform lattice

- *Non-uniform*: $SL(2,\mathbb{R})/\Gamma$ not compact
- *Lattice*: Γ is discrete with $c(\Gamma) \stackrel{def}{=} \text{vol}(SL(2,\mathbb{R})/\Gamma) < \infty$.

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Theorem (Veech '98)

For $f \in B_c(\mathbb{R}^2)$ define
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the Siegel–Veech mean value formula

$$\int_{SL(2,\mathbb{R})/\Gamma} \widehat{f}(g)\, d\mu(g) = \frac{1}{c(\Gamma)} \int_{\mathbb{R}^2} f(x)\, dx.$$

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$$\#\{V \cap B(0,R)\} \sim \frac{1}{c(\Gamma)} \cdot \pi R^2$$

Higher moments for general Γ

Theorem (Fairchild '19)

$$\begin{aligned}
&\int_{SL(2,\mathbb{R})/\Gamma} \left(\widehat{f}\right)^2 (g)\, d\mu(g) \\
&\quad = \frac{1}{c(\Gamma)} \int_{\mathbb{R}^2} f(x)f(x) + f(x)f(-x)\, dx \\
&\qquad + \sum_{n \in N(\Gamma)} \frac{\varphi(n)}{c(\Gamma)} \int_{SL(2,\mathbb{R})} f\left(g \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) f\left(g \begin{bmatrix} 1 \\ n \end{bmatrix}\right) d\eta(g)
\end{aligned}$$

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- (F. 2019) integral formula for $(\widehat{f})^k$ for all $k \in \mathbb{N}$.

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- $N(\Gamma)$ is set of possible determinants.

$$N(\Gamma) = \{n \in \mathbb{R} : \exists \quad v_1, v_2 \in V \text{ s.t. } |v_1 \wedge v_2| = n\}.$$

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1. Maximal parabolic $\Gamma_0 = \mathrm{stab}_{\sigma^{-1}\Gamma\sigma}(e_1) = \left\langle \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \right\rangle$.

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2. $$\varphi(n) = \left| \left\{ \begin{bmatrix} m \\ n \end{bmatrix} \in V : 0 \leq m < h|n| \right\} \right| = \left| \left\{ \Gamma_0 \gamma \Gamma_0 : \gamma = \begin{bmatrix} * & * \\ n & * \end{bmatrix} \in \Gamma \right\} \right|.$$

Sketch of Proof

Theorem (Fairchild '19)

$$\int_{SL(2,\mathbb{R})/\Gamma} \left(\widehat{f}\right)^2(g)\, d\mu(g)$$
$$= \frac{1}{c(\Gamma)} \int_{\mathbb{R}^2} f(x)f(x) + f(x)f(-x)\, dx$$
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Note $\widehat{f} : SL(2,\mathbb{R})/\Gamma \to \mathbb{R}$

$$\widehat{f}(g) = \sum_{v \in V} f(gv)$$

Implies

$$\left(\widehat{f}\right)^2(g) = \sum_{(v_1, v_2) \in V \times V} f(gv_1) f(gv_2)$$

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Decompose $V \times V$ into $SL(2,\mathbb{R})$-orbits:

$$V \times V = \{(v,v) : v \in V\} \sqcup \{(v,-v) : v \in V\} \sqcup \bigsqcup_{n \in N(\Gamma)} \{(v,w) \in V \times V : |v \wedge w| = n\}$$

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Reduction to Γ orbits of D_n

Lemma

$$\int_{SL(2,\mathbb{R})/\Gamma} \sum_{(v_1,v_2)\in D_n} f(gv_1)f(gv_2)\, d\mu(g)$$
$$= \frac{\varphi(n)}{c(\Gamma)} \int_{SL(2,\mathbb{R})} f\left(g\begin{bmatrix}1\\0\end{bmatrix}\right) f\left(g\begin{bmatrix}1\\n\end{bmatrix}\right)\, d\eta(g)$$

For $n \in N(\Gamma)$ define

$$D_n = \{(v,w) \in V \times V : |v \wedge w| = n\}$$

Want to use

$$\int_{SL(2,\mathbb{R})/\Gamma} \sum_{\gamma\in\Gamma} f(g\gamma v_1)f(g\gamma v_2)\, d\mu(g) = \frac{1}{c(\Gamma)} \int_{SL(2,\mathbb{R})} f(gv_1)f(gv_2)\, d\eta(g).$$

φ is number of Γ orbits of D_n

Lemma

$$D_n = \bigsqcup_{\substack{1 \le j \le h|n| \\ (j,n)^T \in V}} \Gamma \cdot \begin{bmatrix} 1 & j \\ 0 & n \end{bmatrix}$$

Thus there are $\varphi(n)$ orbits. Each has a contribution of

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Theorem

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From integrals to asymptotics

$$\sum_{n \in N(\Gamma)} \frac{\varphi(n)}{c(\Gamma)} \int_{SL(2,\mathbb{R})} f\left(g \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) f\left(g \begin{bmatrix} 1 \\ n \end{bmatrix}\right) \, d\eta$$

$$= \frac{1}{c(\Gamma)} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x) f(y) \omega(|x \wedge y|) \, dx \, dy$$

where

$$\omega(t) = \sum_{\substack{n \geq t \\ n \in N(\Gamma)}} \frac{\varphi(n)}{n^3}$$

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Lemma (Good 1983)

$$\sum_{\substack{n \in N(\Gamma) \\ n \leq M}} \varphi(n) = \frac{M^2}{\pi c(\Gamma)} + O(M^{2-\delta})$$

where $0 < \delta < \frac{2}{3}$.

Summary

1. Use general integral formula to gain information about density of pairs of vectors with certain properties
2. Lots of potential in this formula for further understanding. Higher moments too!

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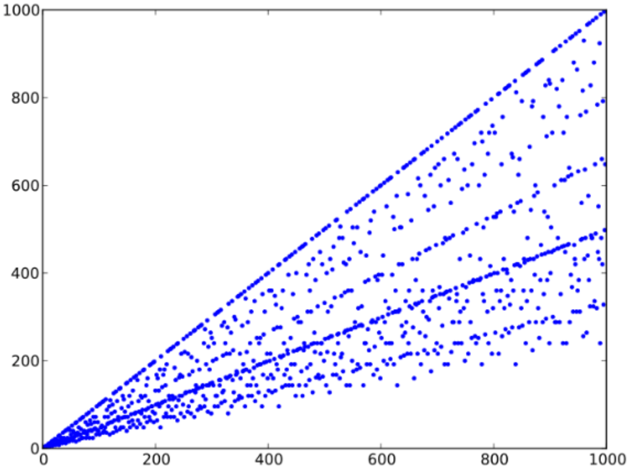
Thank you!

Behavior of φ

- Would like to gain more information about φ
  - $\limsup \frac{\varphi(n)}{n} = 1$?
  - $\liminf \frac{\varphi(n)}{n} = 0$?
  - Use other number theoretic techniques to understand behavior of $\varphi(n)$.



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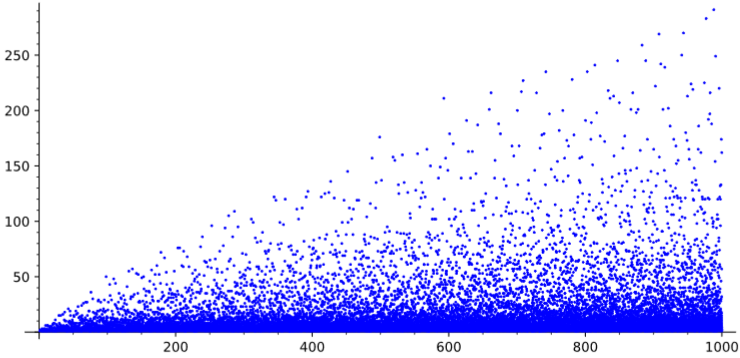

Plot for φ associated to $\Gamma = H_5$ due to Taha '19

φ is not multiplicative for $\Gamma = H_5$

u is the golden ratio

- $\varphi(2u) = 1$
- $\varphi(u) = 2$
- $\varphi(2u^2) = \varphi(2u + 2) = 1.$